ON THE KUDLA-RAPOPORT CONJECTURE FOR UNITARY SHIMURA VARIETIES WITH MAXIMAL PARAHORIC LEVEL STRUCTURE AT UNRAMIFIED PRIMES

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Abstract. In this article, we prove that a version of Tate conjectures for certain Deligne-Lusztig varieties implies the Kudla-Rapoport conjecture for unitary Shimura varieties with maximal parahoric level at unramified primes. Furthermore, we prove that the Kudla-Rapoport conjecture holds unconditionally for several new cases in any dimension.

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1. Introduction

1.1. Introduction.

1.1.1. *Background.* The classical *Siegel–Weil formula* ([Sie35, Sie51, Wei65]) relates special *values* of certain Eisenstein series as theta functions, which are generating series of representation numbers of quadratic forms. Later on, Kudla ([Kud97, Kud04]) proposed an influential program and introduced analogues of theta series in arithmetic geometry. One of the goals of the program is to prove the so-called *arithmetic Siegel–Weil formula* relating the *central derivative* of certain Eisenstein series with a certain arithmetic analogue of theta functions, which is a generating series of arithmetic intersection numbers of n special divisors on Shimura varieties associated to $\mathrm{SO}(n-1,2)$ or $\mathrm{U}(n-1,1)$.

For $\mathrm{U}(n-1,1)$-Shimura varieties, Kudla and Rapoport ([KR11]) formulated a conjectural *local arithmetic Siegel–Weil formula* at an *unramified* place with hyperspecial level, now known as the

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Kudla–Rapoport conjecture. As a local analogue of the arithmetic Siegel-Weil formula, it relates the *central derivative* of local densities of hermitian forms with the arithmetic intersection number of special cycles on unitary Rapoport–Zink spaces. Now this conjectural identity is also known as the Kudla-Rapoport conjecture and was recently proved by Li and Zhang in [LZ22a]. We refer the readers to the introduction of [LZ22a] for more backgrounds and related results.

One of the distinguished features of the hyperspecial case [KR11] is that the corresponding Rapoport-Zink space has good reduction. Accordingly, the analytic side has a clear formulation. A natural and important question is to formulate and prove analogues of the Kudla–Rapoport conjecture when the level structure is non-trivial, where many unexpected new phenomenons occur.

At a ramified place, there are two well-studied unitary Rapoport–Zink spaces with different level structures. One of them is the *exotic smooth model* which has good reduction, and the other one is the *Krämer model* which has bad (semistable) reduction. The analogue of Kudla–Rapoport conjecture for the even dimensional exotic smooth model was formulated and proved by Li and Liu in [LL22] using a strategy similar to [LZ22a]. For the Krämer model, corresponding to the bad reduction in this case, the analytic side is more involved. In fact, even the formulation of the conjecture is not clear and needs to be modified. This phenomenon in the presence of bad reduction was first discovered by Kudla and Rapoport in [KR00] via explicit computation in their study of the Drinfeld p-adic half plane. A similar computation was also done in [San17, HSY20] for unitary special cycles on the Krämer models of the Drinfeld p-adic half plane. The Kudla–Rapoport conjecture for Krämer models, in general, was formulated in [HSY23] with conceptual formulation for the modification and proved in [HLSY23].

The present paper focus on Kudla–Rapoport conjectures with maximal parahoric level structures at an unramified prime, where more new phenomenons occur. If the level structure is almost self-dual, a Kudla-Rapoport type formula was obtained in [San17] when $n = 2$ by explicit computation and established in general in [LZ22a] by relating it with the hyperspecial case. At an unramified prime with general maximal parahoric level structure, such formulation was first given in [Cho22b] in terms of weighted local density taking advantage of a duality between two Rapoport-Zink spaces (see §5). In this paper, we also give another formulation in the spirit of [HSY23, HLSY23] when the intersection number involved is purely contributed by $\mathcal{Z}$-cycles.

Assuming a version of Tate conjectures for certain Deligne-Lusztig varieties, the present paper settles this conjecture for any n and any maximal parahoric level structures. We are able to prove the conjecture unconditionally for several special level structures for any n. In particular, we will give a new proof of the almost self-dual case first proved in [LZ22a]. The main results we obtained should be useful to relax the local assumptions in the arithmetic Siegel–Weil formula for $\mathrm{U}(n-1,1)$–Shimura varieties by allowing maximal parahoric levels at unramified primes. Also, it may be applied to relax the local assumptions in the arithmetic inner product formula in [LL21, LL22] and the p-adic arithmetic inner product formula by Disegni and Liu in [DL22], which are key ingradients for the proof of some special case of the Beilinson-Bloch conjecture. A final remark is that various results obtained in [LZ22a, LL22] are also crucial inputs of the proof of co-rank 1 Siegel-Weil formula for exotic smooth unitary Shimura varieties as in Chen's work [Che24]. Similarly, various results

obtained in the current work should be useful for the co-rank 1 Siegel-Weil formula for unitary Shimura varieties with maximal parahoric levels.

1.1.2. *Kudla–Rapoport conjecture.* Let p be an odd prime. Let F_0 be a finite unramified extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$. Let F be an unramified quadratic extension of F_0. Let π be a uniformizer of both F and F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $O_F, O_{\breve{F}}$ be the ring of integers of $F, \breve{F}$ respectively.

Let $n \geq 2$ be an integer. To define the unitary Rapoport–Zink space with maximal parahoric level structure, we fix a supersingular hermitian O_F-modules $\mathbb{X}$ of signature $(1, n-1)$ over $\bar{k}$. The Rapoport-Zink space $\mathcal{N} = \mathcal{N}_n^{[h]}$ is the formal scheme over $\operatorname{Spf} O_{\breve{F}}$ parameterizing hermitian formal O_F-modules X of signature $(1, n-1)$ and type h (see Definition 2.1) within the quasi-isogeny class of $\mathbb{X}$. The space $\mathcal{N}$ is locally of finite type, and semistable of relative dimension $n-1$ over $\operatorname{Spf} O_{\breve{F}}$.

Let $\overline{\mathbb{E}}$ be the framing hermitian O_F-modules of signature $(0,1)$ over $\bar{k}$. We define *space of quasi-homomorphisms* to be $\mathbb{V} = \mathbb{V}_n \coloneqq \operatorname{Hom}_{O_F}(\overline{\mathbb{E}}, \mathbb{X}) \otimes_{O_F} F$. We can associate $\mathbb{V}$ with a natural F/F_0-hermitian form to make $\mathbb{V}$ a non-degenerate F/F_0-hermitian space of dimension n. For any subset $L \subset \mathbb{V}$, we define the *special cycle* $\mathcal{Z}(L)$ (resp. $\mathcal{Y}(L)$) (see §2.2) to be the deformation locus of L (resp. $\lambda \circ L$) in $\mathcal{N}_n^{[h]}$.

Given an O_F-lattice $L \subset \mathbb{V}$ of full rank n, we can define two integers: the *arithmetic intersection number* $\operatorname{Int}(L)$ and the *modified derived local density* $\partial\operatorname{Den}(L)$.

Definition 1.1. Let $L \subset \mathbb{V}$ be an O_F-lattice and $x_1, \ldots, x_n$ be an O_F-basis of L. We define the *arithmetic intersection number*

$$\operatorname{Int}_{n,h}(L) \coloneqq \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) \in \mathbb{Z}, \tag{1.1}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the special divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and χ denotes the Euler–Poincaré characteristic. By Proposition 2.11, $\operatorname{Int}_{n,h}(L)$ is independent of the choice of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of L itself.

To define the *modified derived local density* $\partial\operatorname{Den}(L)$, we need to introduce local densities first. Let M be another hermitian O_F-lattice (of arbitrary rank) and $\operatorname{Herm}_{L,M}$ denote the O_{F_0}-scheme of hermitian O_F-module homomorphisms from L to M. Then we define the corresponding *local density* to be $\operatorname{Den}(M, L) \coloneqq \lim_{d \to +\infty} \frac{|\operatorname{Herm}_{L,M}(O_{F_0}/\pi^d)|}{q^{d \cdot d_{L,M}}}$, where $d_{L,M}$ is the dimension of $\operatorname{Herm}_{L,M} \otimes_{O_{F_0}} F_0$. Let I_k be an unimodular hermitian O_F-lattice of rank k. It is well-known that there exists a *local density polynomial* $\operatorname{Den}(M, L, X) \in \mathbb{Q}[X]$ such that for any integer $k \geq 0$,

$$\operatorname{Den}(M, L, (-q)^{-k}) = \operatorname{Den}(I_k \oplus M, L). \tag{1.2}$$

Here $I_k \oplus M$ denotes the orthogonal direct sum of I_k and M.

When M also has rank n and $M \otimes_{O_F} F$ is not isometric to $L \otimes_{O_F} F$, we have $\operatorname{Den}(M, L) = 0$. In this case we write

$$\operatorname{Den}'(M, L) \coloneqq -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \operatorname{Den}(M, L, X),$$

and define the (normalized) *derived local density*

$$\mathrm{Den}'_{n,h}(L) \coloneqq \frac{\mathrm{Den}'(I_{n,h}, L)}{\mathrm{Den}(I_{n,h}, I_{n,h})} \in \mathbb{Q}. \tag{1.3}$$

Here $I_{n,h}$ is a hermitian lattice with moment matrix $\mathrm{Diag}((1)^{n-h}, (\pi)^h)$. When the n and h are clear in the context, we also simply denote it as $\mathrm{Den}'(L)$.

Then the naive analogue of the Kudla–Rapoport conjecture for $\mathcal{N}_n^{[h]}$ should be the identity

$$\mathrm{Int}_{n,h}(L) \stackrel{?}{=} \mathrm{Den}'_{n,h}(L).$$

However, as we mentioned before, since $\mathcal{N}_n^{[h]}$ has bad reduction, some modification for the analytic side is needed. Indeed, a similar consideration as in [HSY23, HLSY23] shows that naive analogue cannot be true for trivial reasons. For example, if $h > 0$, then $\mathrm{Int}(L)$ vanishes by definition if L has I_{n-h+1} as a direct summand while $\mathrm{Den}'(L)$ does not vanish for such L by some direct computation. We call an integral O_F-lattice $\Lambda \subset \mathbb{V}$ a *vertex lattice of type* t if $\Lambda^\vee/\Lambda$ is a k-vector space of dimension t. In particular, for a vertex lattice $\Lambda \subset \mathbb{V}$ of type $t < h$, we have $\mathrm{Int}(\Lambda) = 0$, while $\mathrm{Den}'(\Lambda) \neq 0$ in general.

In order to have $\mathrm{Int}(\Lambda_t) = \partial\mathrm{Den}(\Lambda_t)$ for vertex lattice Λ_t of type $t < h$, we define $\partial\mathrm{Den}(L)$ by modifying $\mathrm{Den}'(L)$ with a linear combination of the (normalized) *local densities*

$$\mathrm{Den}_{n,t}(L) \coloneqq \frac{\mathrm{Den}(\Lambda_t, L)}{\mathrm{Den}(\Lambda_t, \Lambda_t)} \in \mathbb{Z}. \tag{1.4}$$

In fact $\mathbb{V} \not\approx I_n^{[h]} \otimes_{O_F} F$. As a result, if $\Lambda_t \subset \mathbb{V}$, then t and h have different parity.

Similarly to the local density polynomial, we define the primitive local density polynomial $\mathrm{Pden}(M, L, X)$ to be the polynomial in $\mathbb{Q}[X]$ such that

$$\mathrm{Pden}(M, L, (-q)^{-k}) \coloneqq \lim_{d \to +\infty} \frac{\#\,\mathrm{Pherm}_{L,M}(O_{F_0}/(\pi^d))}{q^{d \cdot \dim(\mathrm{Herm}_{L,M})_{F_0}}}, \tag{1.5}$$

$$\mathrm{Pherm}_{L,M\oplus H^k}(O_{F_0}/(\pi^d)) \coloneqq \{\phi \in \mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi^d)) \mid \ \phi \text{ is primitive}\}.$$

Recall that $\phi \in \mathrm{Herm}_{L,M\oplus I_k}(O_{F_0}/(\pi^d))$ is primitive if $\dim_{\mathbb{F}_q}((\phi(L) + \pi(M \oplus I_k))/\pi(M \oplus I_k) = n$. In particular, for two lattices L and M of the same rank n, we have

$$\mathrm{Pden}(M, L) = \begin{cases} \mathrm{Den}(M, L) & \text{if } M \cong L, \\ 0 & \text{if } M \ncong L. \end{cases} \tag{1.6}$$

We can also similarly define the normalized primitive local densities:

$$\mathrm{Pden}'_{n,h}(L) = \frac{\mathrm{Pden}'(I_{n,h}, L)}{\mathrm{Den}(I_{n,h}, I_{n,h})}, \qquad \mathrm{Pden}_{n,t}(L) \coloneqq \frac{\mathrm{Pden}(\Lambda_t, L)}{\mathrm{Den}(\Lambda_t, \Lambda_t)},$$

$$\partial\mathrm{Pden}_{n,h}(L) \coloneqq \mathrm{Pden}'_{n,h}(L) + \sum_{i=0}^{\lfloor \frac{(h-1)}{2} \rfloor} c_{n,h-1-2i} \cdot \mathrm{Pden}_{n,h-1-2i}(L).$$

Definition 1.2. Let $L \subset \mathbb{V}$ be an O_F-lattice. Define the *modified derived local density*

$$\partial\mathrm{Den}_{n,h}(L) \coloneqq \mathrm{Den}'_{n,h}(L) + \sum_{i=0}^{\lfloor \frac{(h-1)}{2} \rfloor} c_{n,h-1-2i} \cdot \mathrm{Den}_{n,h-1-2i}(L). \tag{1.7}$$

The coefficients $c_{n,i} \in \mathbb{Q}$ here are chosen to satisfy

$$\partial\mathrm{Den}_{n,h}(\Lambda_i) = 0, \quad \text{for } 0 \leq i \leq h-1 \text{ and } i \equiv h-1 \mod 2, \tag{1.8}$$

which turns out to be a linear system in $(c_{n,i})$ with a unique solution as in [HSY23].

Finally, we propose the following Kudla-Rapoport conjecture for $\mathcal{N}_n^{[h]}$.

Conjecture 1.3 (Conjecture 5.7, Conjecture 6.8)**.** *Let $L \subset \mathbb{V}$ be an O_F-lattice. Then we have*

$$\mathrm{Int}_{n,h}(L) = \partial\mathrm{Den}_{n,h}(L).$$

Remark 1.4. Conjecture 5.7 is a special case of the conjecture formulated in [Cho22b], which follows a different philosophy. However, we show that Conjecture 1.3 is equivalent to Conjecture 5.7 by Corollary 6.7 and Proposition 6.9.

Remark 1.5. We remark that $\mathrm{Den}'_{n,h}(L)$ is not an integer in general. Only the modified $\partial\mathrm{Den}_{n,h}(L)$ is an integer. However, a priori, this is not clear at all.

The main purpose of this paper is to prove Conjecture 1.3 assuming a version of Tate conjectures for certain Deligne-Lusztig varieties.

Theorem 1.6 (Theorem 9.4)**.** *Let $L \subset \mathbb{V}$ be an O_F-lattice of rank n. Assuming Conjecture 4.3, we have*

$$\mathrm{Int}_{n,h}(L) = \partial\mathrm{Den}_{n,h}(L).$$

In this paper, we also verified Conjecture 4.3 for $\mathcal{N}_n^{[1]}$, $\mathcal{N}_n^{[n-1]}$ and $\mathcal{N}_4^{[2]}$ unconditionally.

Theorem 1.7 (Theorem 4.5, Theorem 9.5)**.** *Assume (n,h) is one of the following cases: $(n,1),(n,n-1)$ and $(4,2)$. Then Conjecture 1.3 holds unconditionally.*

Remark 1.8. Conjecture 1.3 for $\mathcal{N}_n^{[0]}$ was the original Kudla-Rapoport conjecture proposed in [KR11] and proved in [LZ22a]. The case $(n,h) = (n,1)$ was first proved by [LZ22a] using certain Hecke correspondence that relates $\mathcal{N}_n^{[1]}$ with $\mathcal{N}_{n+1}$. Our proof is an attempt to prove the general case uniformly in a way closer to the proof for $\mathcal{N}_n^{[0]}$ in [LZ22a].

Remark 1.9. Although $\mathcal{N}_n^{[1]} \cong \mathcal{N}_n^{[n-1]}$ by a natural duality, $\mathcal{Z}$-cycles on $\mathcal{N}_n^{[1]}$ will be transformed into $\mathcal{Y}$-cycles on $\mathcal{N}_n^{[n-1]}$. Hence, Theorem 1.7 for $\mathcal{N}_n^{[n-1]}$ is different with the case for $\mathcal{N}_n^{[1]}$. In particular, Theorem 1.7 for $\mathcal{N}_n^{[n-1]}$ and $\mathcal{N}_4^{[2]}$ are new.

Remark 1.10. Conjecture 1.3 is formulated based on a different viewpoint from the one formulated in [Cho22b]. The conjecture formulated in [Cho22b] is inspired by the duality $\mathcal{N}_n^{[h]} \cong \mathcal{N}_n^{[n-h]}$ and is more general in the sense that it also considers the case when the intersection is between $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles. However, since our main theorem in this paper is about intersections between $\mathcal{Z}$-cycles and the above formulation is closer to previously studied cases e.g.

[LZ22a],[LL22],[LZ22b],[HLSY23], we state this new formulation in the introduction. We refer the reader to Conjecture 5.6 for the general conjecture formulated in [Cho22b] and Conjecture 5.7 for the specialized conjecture about the intersection between $\mathcal{Z}$-cycles. We show that conjectures 1.3 and 5.7 are in fact equivalent in Proposition 6.9, which is interestingly a nontrivial fact to prove.

1.1.3. *Strategy and novelty.* Overall, our general strategy is similar to the strategy of [LZ22a] using the local modularity and uncertainty principle, although we need to overcome various new difficulties due to the parahoric levels. More precisely, fix an O_F-lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$ and consider functions on $\mathbb{V} \setminus L_F^\flat$,

$$\mathrm{Int}_{L^\flat}(x) \coloneqq \mathrm{Int}(L^\flat + \langle x \rangle), \quad \partial\mathrm{Den}_{L^\flat}(x) \coloneqq \partial\mathrm{Den}(L^\flat + \langle x \rangle).$$

We need to show $\mathrm{Int}_{L^\flat} = \partial\mathrm{Den}_{L^\flat}$ as functions on $\mathbb{V} \setminus L_F^\flat$. First, we have a natural decomposition of $\mathrm{Int}_{L^\flat}$ and $\partial\mathrm{Den}_{L^\flat}$ into horizontal parts and vertical parts:

$$\mathrm{Int}_{L^\flat} = \mathrm{Int}_{L^\flat,\mathscr{H}} + \mathrm{Int}_{L^\flat,\mathscr{V}}, \quad \partial\mathrm{Den}_{L^\flat} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}} + \partial\mathrm{Den}_{L^\flat,\mathscr{V}}.$$

For the horizontal parts, we have $\mathrm{Int}_{L^\flat,\mathscr{H}} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ by direct comparison. For the vertical parts, $\mathrm{Int}_{L^\flat,\mathscr{V}}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ satisfy "*local modularity*" in the sense that their Fourier-transforms have nice behaviors.

Due to the existence of nontrivial levels, new phenomenons have already shown up for the horizontal part. We can decompose the horizontal cycles into primitive horizontal cycles indexed by "horizontal lattices" and essentially reduce the computation to $n = 2$, similar to [LZ22a]. However, first of all, we have two different types of the "horizontal lattices" indexing the primitive horizontal cycle. Moreover, the primitive piece indexed by one of them admits a further decomposition into mixed special cycles. Here mixed special cycles mean that they are intersections between both $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles. We refer the readers to Theorem 3.3 for more details.

Now we turn our attention to the vertical part. We discuss the geometric side first. For a curve $C \subset \mathcal{N}_n^{[h]}$, let $\mathrm{Int}_C(\mathcal{Z}(x)) \coloneqq \chi(\mathcal{N}_n^{[h]}, \mathcal{O}_C \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)})$ and $\mathrm{Int}_C(\mathcal{Y}(x)) \coloneqq \chi(\mathcal{N}_n^{[h]}, \mathcal{O}_C \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Y}(x)})$. If C lies in $\mathcal{N}_2^{[1]}$ or $\mathcal{N}_3^{[0]}$ and is embedded into $\mathcal{N}_n^{[h]}$ via an embedding of $\mathcal{N}_2^{[1]}$ or $\mathcal{N}_3^{[0]}$ into $\mathcal{N}_n^{[h]}$, then explicit computation shows that

$$\widehat{\mathrm{Int}}_C(\mathcal{Z}(x)) = -q^{-h} \cdot \mathrm{Int}_C(\mathcal{Y}(x)), \tag{1.9}$$

where $\widehat{\mathrm{Int}}_C(\mathcal{Z}(x))$ denotes the Fourier transform of $\mathrm{Int}_C(\mathcal{Z}(x))$ (see §1.2 for more details). We call such a curve C a special curve. This was first observed in [LZ22a] when $h = 0$. In fact, when $h = 0$, the more general identity

$$\widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}}(\mathcal{Z}(x)) = -\mathrm{Int}_{L^\flat,\mathscr{V}}(\mathcal{Y}(x))$$

was proved in [LZ22a], which we call *local modularity* in this case. When $h \geq 0$ and C is special, (1.9) was observed in [Zha22] based on computation of intersection numbers on $\mathcal{N}_2^{[1]}$ established in [San17]. In [Zha22], this local modularity was used to prove arithmetic transfer conjecture in a similar setting.

Nevertheless, it is by no means clear that this will be true in general. One observation is that if $\mathrm{Int}_{L^\flat,\mathscr{V}}(x)$ can be written as a linear combination of $\mathrm{Int}_C(x)$ for special C, then we have

$$\text{(1.10)}\qquad \text{“}\widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}}(x)\text{“} = \text{”} \begin{cases} -q^{-h}\cdot \mathrm{Int}_{n-1,h-1,\mathscr{V}}(L^\flat) & \text{if } \mathrm{val}(x)=-1,\\ 0 & \text{if } \mathrm{val}(x)\leq -2. \end{cases}$$

Although it is not clear whether $\mathrm{Int}_{L^\flat,\mathscr{V}}(x)$ can be written as a linear combination of $\mathrm{Int}_C(x)$ for special C, one can always try to test the corresponding speculation on the analytic side, which is a central idea for many of the past works. Indeed, for example, we recognized that there should exist two different types of horizontal lattices by testing the analytic side first (the horizontal part goes to infinity when $\mathrm{val}(x)$ goes to infinity). Also, it is really the computation of $\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{V}}(x)$ in the $\mathcal{N}_4^{[2]}$ case inspires us what we should expect in general for the geometric side.

On the other hand, once we have a big picture for the geometric side with the help of explicit computation from the analytic side, the insight from the geometric side also serves as a guiding principle to obtain purely analytic results. For example, [HLSY23, Proposition 7.5] is inspired by the fact that $\mathcal{Z}(L)$ is empty for non-integral L, and the proof of [HLSY23, Proposition 7.5] gives important hints for how to prove the rest major analytic results of [HLSY23]. This is also what happened in the current case.

Indeed, as one of the main results, we manage to prove an analytic analogue of (1.10) unconditionally. More precisely, we prove $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x)$ can be extended to $\mathbb{V}$ as a locally constant function and

$$\text{(1.11)}\qquad \widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{V}}(x) = \begin{cases} -q^{-h}\cdot \mathrm{Int}_{n-1,h-1,\mathscr{V}}(L^\flat) & \text{if } \mathrm{val}(x)=-1,\\ 0 & \text{if } \mathrm{val}(x)\leq -2. \end{cases}$$

As a result, assuming (1.10), for any x with $\mathrm{val}(x)<0$, we have

$$\widehat{\mathrm{Int}}_{L^\flat}(x) - \widehat{\partial\mathrm{Den}}_{L^\flat}(x) = \widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}}(x) - \widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{V}}(x) = 0.$$

Then the identity $\mathrm{Int}_{L^\flat}(x) - \partial\mathrm{Den}_{L^\flat}(x) = 0$ follows from the uncertainty principle as in [LZ22a].

We remark that (1.11) suggests that $\mathrm{Int}_{L^\flat,\mathscr{V}}(x)$ might be written as a linear combination of $\mathrm{Int}_C(x)$ for special curve C. When $h=0$, this is indeed the case and was proved in [LZ22a, Corollary 5.3.3]. Therefore we propose this statement as a conjecture in Conjecture 4.3.

In order to establish (1.11), we make use of the *primitive decomposition* of the local density polynomial into primitive local density polynomials and obtain a decomposition of $\partial\mathrm{Den}(L)$:

$$\text{(1.12)}\qquad \partial\mathrm{Den}(L) = \sum_{L\subset L'} \partial\mathrm{Pden}(L'),$$

where L' runs over O_F-lattices in L_F containing L, and the symbol Pden stands for the primitive version of Den (see (1.5)). The key reason to consider this primitive decomposition and $\partial\mathrm{Pden}(L)$ is that we usually have a very simple formula for $\partial\mathrm{Pden}(L)$ which eventually makes the computation about $\partial\mathrm{Den}(L)$ possible. This is the case for all the previously proved Kudla-Rapoport type formulas, e.g. [LZ22a],[LL22],[LZ22b], and [HLSY23].

Although a similar approach to compute $\partial\mathrm{Pden}(L)$ as in [HLSY23] may be directly generalized to the current situation, we decide to generalize the main result of [Cho23] to obtain an explicit formula

of $\partial\mathrm{Pden}(L)$ (see Proposition 6.17). One of the reasons we choose this approach is due to the fact that Proposition 6.17 itself is already a certain induction formula that reduces the computation of $\partial\mathrm{Pden}(L)$ to the good reduction case (see Remark 6.18), which is particularly handy in order to prove Theorem 1.11 below.

One of the major *new phenomenons and difficulties* we found and overcame in this paper is the fact that when there is a non-trivial level structure, $\partial\mathrm{Pden}(L)$ *no longer* has a simple formula in general compared with the previously proved cases. This can be seen via some explicit computation using Proposition 6.17. Nevertheless, inspired by an attempt to compute the Fourier transform of $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x)$ (e.g. to obtain (1.11)), we find simple inductive formulas for $\partial\mathrm{Pden}(L)$ which suffices to control the Fourier transform of $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x)$. In fact, such inductive formulas also hold in all the previously studied cases and the simple formulas for $\partial\mathrm{Pden}(L)$ in these cases follow as a direct corollary.

More precisely, let $t_i(L)$ and $t_{\geq i}(L)$ be the number of the fundamental invariants of L that is exactly i and at least i respectively. Then we have the following.

Theorem 1.11 (Theorem 7.4)**.** *Let L be a hermitian lattice with $\mathrm{val}(L) \not\equiv h \mod 2$. We have $\partial\mathrm{Pden}_{n,h}(L)$ depends only on $(t_{\geq 2}(L), t_1(L), t_0(L))$. For simplicity, we denote it as $D_{n,h}(t_2, t_1, t_0)$. Moreover, assume that $(t_2 - 1, t_1 + 1, t_0) \neq (n-h, h, 0)$, $t_0 \leq n-h$ and $t_2 \geq 1$, then we have*

$$D_{n,h}(t_2, t_1, t_0) - D_{n,h}(t_2 - 1, t_1 + 1, t_0) = -(-q)^{2n-h-1-t_1-2t_0} D_{n-1,h-1}(t_2 - 1, t_1, t_0).$$

The above inductive formula is complemented by the following simple formulas for special $D_{n,h}(t_2, t_1, t_0)$.

Theorem 1.12. *With the same notations and assumptions as in Theorem 1.11, we have the following.*

(1) (Lemma 7.5, Proposition 7.6) If $t_0 > n - h$, then $D_{n,h}(t_2, t_1, t_0) = 0$.

(2) (Theorem 7.8 (1)) If $t_2 = 0$ and $h + 1 \leq t_1$. Then, we have

$$D_{n,h}(0, t_1, t_0) = \frac{\prod_{l=h+1}^{t_1}(1 - (-q)^l)}{(1 - (-q)^{t_1 - h})}.$$

(3) (Theorem 7.8 (2)) If $t_2 = 1$ and $h - 1 \leq t_1$. Then, we have

$$D_{n,h}(1, t_1, t_0) = \begin{cases} 1 & \text{if } t_1 = h-1, h; \\ \prod_{l=h+1}^{t_1}(1 - (-q)^l) & \text{if } t_1 \geq h+1. \end{cases}$$

Theorems 1.11 and 1.12 will be proved in §7 using the formula for $\partial\mathrm{Pden}(L)$ obtained in [Cho23]. As we have remarked, the method developed in [HLSY23] may also be adapted to the current case. However, we found that using the formulas in [Cho23] is easier for our purpose so we stick with this approach. Note that the formulas in [Cho23] are derived via a duality between weighted local densities in analogy with the duality between $\mathcal{N}_n^{[h]}$ and $\mathcal{N}_n^{[n-h]}$, and this might be a particular reason for why this formula so applicable in the current case.

With the help of formulas for $\partial\mathrm{Pden}(L)$, we finally prove (1.11) via certain involved weighted lattice counting in §8 via a similar method as in [LZ22b, HLSY23]. However, since $\partial\mathrm{Pden}(L)$ now depends on $t_{\geq 2}(L), t_1(L)$ and $t_0(L)$, the counting becomes much more involved compared with the ones in [LZ22b, HLSY23].

1.2. **Notation and terminology.**

- Let p be an odd prime. Let F_0 be a finite unramified extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$. Let F be the unramified quadratic extension of F_0. Let π be a uniformizer of F and F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $O_F, O_{\breve{F}}$ be the ring of integers of $F, \breve{F}$ respectively.
- We say a sublattice of a hermitian space is non-degenerate if the restriction of the hermitian form to it is non-degenerate.
- In this paper, without explicit mentioning a lattice means a non-degenerate hermitian O_F-lattice. Unless otherwise stated, the symbol L always means a non-degenerate lattice of rank n with a hermitian form $(\ ,\ )$.
- We define $L^\vee$ to be the dual lattice of L with respect to the hermitian form $(\ ,\ )$. If $L \subset L^\vee$, we say L is integral. If $L \subset L^\vee \subset \pi^{-1}L$, we say L is a vertex lattice.
- We say that a basis $\{\ell_1, \dots, \ell_n\}$ of L is a normal basis (which always exists) if its moment matrix $T = ((\ell_i, \ell_j))_{1\le i,j\le n}$ is $\mathrm{Diag}(\pi^{\alpha_1}, \dots, \pi^{\alpha_n})$, where $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$. Moreover, we define its fundamental invariants $(a_1, \cdots, a_n)$ to be the unique nondecreasing rearrangement of $(\alpha_1, \dots, \alpha_n)$.
- We define the valuation of L to be $\mathrm{val}(L) \coloneqq \sum_{i=1}^n a_i$, where $(a_1, \cdots, a_n)$ are the fundamental invariants of L. For $x \in L$, we define $\mathrm{val}(x) = \mathrm{val}((x, x))$, where $\mathrm{val}(\pi) = 1$.
- We call a sublattice $N \subset M$ primitive in M if $\dim_{\mathbb{F}_q} \overline{N} = r(N)$, where $\overline{N} = (N + \pi M)/\pi M$. We also use $\overline{L}$ to denote $L \otimes_{O_F} O_F/(\pi)$.
- We use I_m to denote a unimodular lattice of rank m, and $I_{n,h}$ to denote a hermitian lattice with moment matrix $\mathrm{Diag}((1)^{n-h}, (\pi)^h)$.
- For a hermitian space $\mathbb{V}$, we let $\mathbb{V}^{?i} \coloneqq \{x \in \mathbb{V} \mid \mathrm{val}(x) ? i\}$ where ? can be $\ge$, $\le$ or $=$.
- Fix an unramified additive character $\psi : F_0 \to \mathbb{C}^\times$. Here "unramifiedness" means that the conductor of ψ (i.e., the largest fractional ideal in F_0 on which ψ is trivial) is O_{F_0}. For an integrable function f on $\mathbb{V}$, we define its Fourier transform $\widehat{f}$ to be

$$\widehat{f}(x) \coloneqq \int_{\mathbb{V}} f(y)\psi(\mathrm{tr}_{F/F_0}(x, y))\mathrm{d}y, \quad x \in \mathbb{V}.$$

  We normalize the Haar measure on $\mathbb{V}$ to be self-dual, so $\hat{f}(x) = f(-x)$. For an O_F-lattice $L \subset \mathbb{V}$ of rank n, we have (under the assumption that F/F_0 is unramified)

$$\widehat{\mathbf{1}}_L = \mathrm{vol}(L)\mathbf{1}_{L^\vee}, \quad \text{and} \quad \mathrm{vol}(L) = \left[L^\vee : L\right]^{-1/2} = q^{-\mathrm{val}(L)}.$$

  Note that $\mathrm{val}(L)$ can be defined for any lattice L (not necessarily integral) so that the above equality for $\mathrm{vol}(L)$ holds.

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2. Rapoport-Zink space and special cycles

2.1. **Rapoport-Zink space.** In this section, we review the definition and basic properties of Rapoport-Zink space and special cycles.

Definition 2.1. For any $\operatorname{Spf} O_{\breve{F}}$-scheme S, *a hermitian formal O_F-moodule* (X, ι, λ) of signature $(1, n-1)$ and type h over S is the following data:

- X is a strict formal O_{F_0}-module over S of relative height $2n$ and dimension n. Strictness means the induced action of O_{F_0} on $\operatorname{Lie} X$ is via the structure morphism $O_{F_0} \to \mathcal{O}_S$.
- $\iota : O_F \to \operatorname{End}(X)$ is an action of O_F on X that extends the action of O_{F_0}. We require that the *Kottwitz condition* of signature $(1, n-1)$ holds for all $a \in O_F$:

$$\operatorname{char}(\iota(a) \mid \operatorname{Lie} X) = (T-a)(T-\overline{a})^{n-1} \in \mathcal{O}_S[T]. \tag{2.1}$$

- λ is a polarization on X, which is O_F/O_{F_0} semi-linear in the sense that the Rosati involution $\operatorname{Ros}_\lambda$ induces the non-hrivial involution on $\iota : O_F \to \operatorname{End}(X)$.
- We require that the finite flat group scheme $\operatorname{Ker} \lambda$ over S lies in $X[\pi]$ and is of order q^{2h}.

An isomorphism $(X_1, \iota_1, \lambda_1) \xrightarrow{\sim} (X_2, \iota_2, \lambda_2)$ between two such triples is an O_F-linear isomorphism $\varphi\colon X_1 \xrightarrow{\sim} X_2$ such that $\varphi^*(\lambda_2) = \lambda_1$. Up to O_F-linear quasi-isogeny compatible with the polarization, there exists a unique such triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\mathbb{F}$. Fix one choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as the *framing object*.

Definition 2.2. Let (Nilp) be the category of O_F-schemes S such that π is locally nilpotent on S. Then the Rapoport–Zink space associated with $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is the functor

$$\mathcal{N}_{(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})} = \mathcal{N}_n^{[h]} \to \operatorname{Spf} O_{\breve{F}}$$

sending $S \in$ (Nilp) to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$, where

- (X, ι, λ) is a hermitian formal O_F-module of dimension n and type h over S;
- $\rho : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}} \overline{S}$ is an O_F-linear quasi-isogeny of height 0 over the reduction $\overline{S} := S \times_{O_{\breve{F}_0}} \mathbb{F}$ such that $\rho^*(\lambda_{\mathbb{X}, \overline{S}}) = \lambda_{\overline{S}}$.

The functor $\mathcal{N}_n^{[h]}$ is representable by a formal scheme over $\operatorname{Spf} O_{\breve{F}}$ which is locally formally of finite type by [RZ96]. Moreover, this formal scheme is regular (see [Cho18, Proposition 3.33]). We often simply denote $\mathcal{N}_n^{[h]}$ as $\mathcal{N}$ if the signature and type are clear in the context.

There is an isomorphism $\theta : \mathcal{N}_n^{[h]} \xrightarrow{\sim} \mathcal{N}_n^{[n-h]}$ constructed as follows ([Cho18, Remark 5.2]). For each $S \in$ (Nilp),

$$\mathcal{N}_n^{[h]}(S) \xrightarrow{\theta} \mathcal{N}_n^{[n-h]}(S),$$
$$(X, i_X, \lambda_X, \rho_X) \mapsto (X^\vee, \bar{i}_X^\vee, \lambda_X', (\rho_X^\vee)^{-1}),$$

where $\lambda_X' : X^\vee \to X$ is the unique polarization such that $\lambda_X' \circ \lambda_X = i_X(\pi)$, and for $a \in O_F$, the action $\bar{i}_X^\vee$ is defined as $\bar{i}_X^\vee(a) := i_X(a^*)^\vee$. When we denote $\mathcal{N}_n^{[h]}$ as $\mathcal{N}$, we use $\mathcal{N}^\vee$ to denote $\mathcal{N}_n^{[n-h]}$.

2.2. **Special cycles.** To define special cycles, we need to fix a hermitian formal O_F-moodule $(\overline{\mathbb{E}}, i_{\overline{\mathbb{E}}}, \lambda_{\overline{\mathbb{E}}})$ of signature $(0,1)$ and type 0 over $\mathbb{F}$. Then we can similarly define a Rapoport-Zink Space $\mathcal{N}_{(\overline{\mathbb{E}}, i_{\overline{\mathbb{E}}}, \lambda_{\overline{\mathbb{E}}})}$ which we denote as $\mathcal{N}^0$ for simplicity. Recall that there is a unique lifting $(\overline{\mathcal{E}}, \iota_{\overline{\mathcal{E}}}, \lambda_{\overline{\mathcal{E}}})$ of $(\overline{\mathbb{E}}, i_{\overline{\mathbb{E}}}, \lambda_{\overline{\mathbb{E}}})$ over $O_{\breve{F}}$.

Definition 2.3. We define the space of special homomorphism to be the F-vector space

$$\mathbb{V} \coloneqq \mathrm{Hom}_{O_F}(\overline{\mathbb{E}}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We can associate $\mathbb{V}$ with a naturally defined hermitian form as follows. For $x, y \in \mathbb{V}$, we define a hermitian form h on $\mathbb{V}$ as

$$(x,y) = \lambda_{\overline{\mathbb{E}}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{O_F}(\overline{\mathbb{E}}) \otimes \mathbb{Q} \overset{i_{\overline{\mathbb{X}}}^{\simeq}}{\simeq} F.$$

We often omit $i_{\overline{\mathbb{E}}}^{-1}$ via the identification $\mathrm{End}_{O_F}(\overline{\mathbb{E}}) \otimes \mathbb{Q} \simeq F$.

Definition 2.4. [KR11, Definition 3.2], [Cho18, Definition 5.4]

(1) For $x \in \mathbb{V}$, we define the special cycle $\mathcal{Z}(x)$ to be the closed formal subscheme of $\mathcal{N}^0 \times \mathcal{N}$ with the following property: For each O_F-scheme S such that π is locally nilpotent, $\mathcal{Z}(x)(S)$ is the set of all points $\xi = (\overline{\mathcal{E}}_S, \iota_{\overline{\mathcal{E}}_S}, \lambda_{\overline{\mathcal{E}}_S}, X, i_X, \lambda_X, \rho_X)$ in $\mathcal{N}^0 \times \mathcal{N})(S)$ such that the quasi-homomorphism

$$\rho_X^{-1} \circ x \circ \rho_{\overline{\mathcal{E}}_S} : \overline{\mathcal{E}}_S \times_S \overline{S} \to X \times_S \overline{S}$$

extends to a homomorphism from $\overline{\mathcal{E}}_S$ to X.

(2) For $y \in \mathbb{V}$, we define the special cycle $\mathcal{Y}(y)$ in $\mathcal{N}^0 \times \mathcal{N}$ as follows. First, consider the special cycle $\mathcal{Z}(\lambda_{\mathbb{X}} \circ y)$ in $\mathcal{N}^0 \times \mathcal{N}^\vee$. This is the closed formal subscheme of $\mathcal{N}^0 \times \mathcal{N}^\vee$. We define $\mathcal{Y}(y)$ as $(id \times \theta^{-1})(\mathcal{Z}(\lambda_{\mathbb{X}} \circ y))$ in $\mathcal{N}^0 \times \mathcal{N}$

Note that $\mathcal{N}^0$ can be identified with $\mathrm{Spf}\, O_{\breve{F}}$, and hence $\mathcal{Z}(x), \mathcal{Y}(y)$ can be identified with closed formal subschemes of $\mathcal{N}$. The same proof of [KR14, Proposition 5.9] gives us the following.

Proposition 2.5. [KR14, Proposition 5.9] *The functors $\mathcal{Z}(x)$ and $\mathcal{Y}(y)$ are representable by Cartier divisors of $\mathcal{N}$.*

Proposition 2.6. [Cho18, Proposition 5.10] *Let $x, y \in \mathbb{V}$.*

(1) If $\mathrm{val}((x,x)) = 0$, then $\mathcal{Z}(x) \simeq \mathcal{N}_{n-1}^{[h]}$.

(2) If $\mathrm{val}((y,y)) = -1$, then $\mathcal{Y}(y) \simeq \mathcal{N}_{n-1}^{[h-1]}$.

2.3. **Horizontal and vertical part of special cycles.** We closely follow [LZ22a, §2.9] in this subsection. We call a formal scheme Z over $\mathrm{Spf}\, O_{\breve{F}}$ vertical (resp. horizontal) if π is locally nilpotent on Z (resp. flat over $\mathrm{Spf}\, O_{\breve{F}}$). In particular, the formal scheme-theoretic union of two vertical (resp. horizontal) formal subschemes of a formal scheme is again vertical (resp. horizontal).

Now we define the horizontal part and vertical part of Z respectively. The horizontal part $Z_{\mathscr{H}}$ of Z is defined to be the closed formal subscheme with ideal sheaf $\mathcal{O}_Z[\pi^\infty] \subset \mathcal{O}_Z$. Then $Z_{\mathscr{H}}$ is the maximal horizontal closed formal subscheme of Z. For noetherian Z, we can find $N \gg 0$ such that $\pi^N \mathcal{O}_Z[\pi^\infty] = 0$. Then the vertical part $Z_{\mathscr{V}} \subset Z$ is defined to be the closed formal subscheme with ideal sheaf $\pi^N \mathcal{O}_Z$. Note that $\mathcal{O}_Z[\pi^\infty] \cap \pi^N \mathcal{O}_Z = 0$ implies the following decomposition:$Z = Z_{\mathscr{H}} \cup Z_{\mathscr{V}}$. The same proof of [LZ22a, Lemma 2.9.2] gives the following.

Lemma 2.7. [LZ22a, Lemma 2.9.2] *Let $L \subset \mathbb{V}$ be a O_F-lattice of rank $r \geq n-1$ such that L_F is non-degenerate. Then $\mathcal{Z}(L)$ is noetherian.*

The following lemma follows from the same proof of [LZ22a, Lemma 5.1.1].

Lemma 2.8. [LZ22a, Lemma 5.1.1] *Let $L^\flat$ be an O_F-lattice of rank $n-1$ in $\mathbb{V}_n$. Then $\mathcal{Z}(L^b)_{\mathscr{V}}$ is supported on $\mathcal{N}_n^{\mathrm{red}}$, i.e., $\mathcal{O}_{\mathcal{Z}(L^b)_{\mathscr{V}}}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}_n^{\mathrm{red}} \subset \mathcal{N}_n$.*

2.4. **Linear invariance.** Following [How19], we show the linear invariance of intersection numbers. In this subsection, we use $\mathcal{N}$ to denote $\mathcal{N}_n^{[h]}$ and (X, ι_X, λ_X) to denote the the universal object over $\mathcal{N}$. Let $\mathrm{D}(X)$ denote the covariant Grothendieck-Messing crystal of X restricted to the Zariski site. Then we have a short exact sequence of locally free $\mathcal{O}_{\mathcal{N}}$-modules:

$$0 \to \mathrm{Fil}(X) \to \mathrm{D}(X) \to \mathrm{Lie}(X) \to 0.$$

which is O_F-linear via the action given by ι_X. Let $O_F = O_{F_0} + O_{F_0}\alpha$ and

$$\epsilon := \bar{\alpha} \otimes 1 - 1 \otimes \bar{\alpha} \in O_F \otimes_{O_{F_0}} \mathcal{O}_{\mathcal{N}},$$
$$\bar{\epsilon} := \alpha \otimes 1 - 1 \otimes \bar{\alpha} \in O_F \otimes_{O_{F_0}} \mathcal{O}_{\mathcal{N}}.$$

Definition 2.9. Let L_X be the image of ϵ on $\mathrm{Lie}(X)$. In other words, $L_X := \epsilon\,\mathrm{Lie}(X)$.

According to the Kottwitz signature condition, we know $L_X \subset \mathrm{Lie}(X)$ is locally a $\mathcal{O}_{\mathcal{N}}$-module direct summand of rank 1.

For a closed formal subscheme Z of $\mathcal{N}$ with ideal sheaf $\mathcal{I}_Z$, we denote by $\tilde{Z}$ the closed formal subscheme defined by the sheaf $\mathcal{I}_Z^2$. Let $x \in \mathbb{V}$ be a non-zero special homomorphism. Let X_0 be the universal object of $\mathcal{N}_1^{[0]}$. By the very definition of $\mathcal{Z}(x)$, we have

$$X_0|_{\mathcal{Z}(x)} \xrightarrow{x} X|_{\mathcal{Z}(x)},$$

which induces an O_F-linear morphism of vector bundles

$$\mathrm{D}(X_0)|_{\mathcal{Z}(x)} \xrightarrow{x} \mathrm{D}(X)|_{\mathcal{Z}(x)}.$$

By the Grothendieck–Messing theory, we may canonically extend the above morphism to a morphism

$$\mathrm{D}(X_0)|_{\tilde{\mathcal{Z}}(x)} \xrightarrow{\tilde{x}} \mathrm{D}(X)|_{\tilde{\mathcal{Z}}(x)},$$

which no longer preserves the Hodge filtrations and hence induces a nontrivial morphism

$$\mathrm{Fil}(X_0)|_{\tilde{\mathcal{Z}}(x)} \xrightarrow{\tilde{x}} \mathrm{Lie}(X)|_{\tilde{\mathcal{Z}}(x)}. \tag{2.2}$$

Proposition 2.10. *The morphism (2.2) induces a morphism*

$$\mathrm{Fil}(X_0)|_{\tilde{Z}(x)} \xrightarrow{\tilde{x}} L_X|_{\tilde{Z}(x)}.$$

Moreover, $Z(x)$ is the vanishing locus of $\tilde{x}$.

Proof. According to the signature condition of X_0, we have $\mathrm{Fil}(X_0) = \epsilon \mathrm{D}(X_0)$ since both are locally $\mathcal{O}_{\mathcal{N}}$ direct summands of $\mathrm{D}(X_0)$ of rank 1 where $\iota(a)$ acts as $\bar{a}$. Since $\tilde{x}$ is O_F-linear, we have

$$\tilde{x}(\mathrm{Fil}(X_0)) = \tilde{x}(\epsilon \mathrm{D}(X_0)) \in \epsilon\,\mathrm{Lie}(X) = L_X.$$

Now the second claim follows from the Grothendieck–Messing theory. □

Now given a nonzero element $x \in \mathbb{V}$, we define a chain complex of locally free $\mathcal{O}_{\mathcal{N}}$-modules

$$C(x) \coloneqq (\cdots \to 0 \to \mathcal{I}_{\mathcal{Z}(x)} \to \mathcal{O}_{\mathcal{N}} \to 0)$$

supported in degrees 1 and 0 with the map $\mathcal{I}_{\mathcal{Z}(x)} \to \mathcal{O}_{\mathcal{N}}$ being the natural inclusion. We extend the definition to $x = 0$ by setting

$$C(0) \coloneqq (\cdots \to 0 \to \omega \xrightarrow{0} \mathcal{O}_{\mathcal{N}} \to 0)$$

supported in degrees 1 and 0 , where ω is the line bundle such that $\omega^{-1} = \underline{\mathrm{Hom}}(\mathrm{Fil}(X_0), L_X)$.

The following is our main result of this subsection which follows from Proposition 2.11 by the same argument as in [How19].

Proposition 2.11. *Let $0 \leqslant m \leqslant n$ be an integer. Suppose that $x_1, \ldots, x_m \in \mathbb{V}$ and $y_1, \ldots, y_m \in \mathbb{V}$ generate the same O_E-submodule. Then we have an isomorphism*

$$\mathrm{H}_i(C(x_1) \otimes_{\mathcal{O}_{\mathcal{N}}} \cdots \otimes_{\mathcal{O}_{\mathcal{N}}} C(x_m)) \simeq \mathrm{H}_i(C(y_1) \otimes_{\mathcal{O}_{\mathcal{N}}} \cdots \otimes_{\mathcal{O}_{\mathcal{N}}} C(y_m))$$

of $\mathcal{O}_{\mathcal{N}}$-modules for every i.

3. Horizontal parts of Kudla–Rapoport cycles

In this section, we describe the horizontal parts of special cycles following the approach of [LZ22a, §4]. Due to the existence of non-trivial level structures, there are some new phenomena.

Let K denote a finite extension of $\breve{F}$. Consider $z \in \mathcal{N}_n^{[h]}(O_K)$ which corresponds to an O_F-hermitian module G of signature $(1, n-1)$ over O_K. Let $T_p(-)$ denote the integral p-adic Tate modules and

$$L \coloneqq \mathrm{Hom}_{O_F}(T_p\overline{\mathcal{E}}, T_pG).$$

We can associate L with a hermitian form $\{x, y\}$ given by

$$(T_p\overline{\mathcal{E}} \xrightarrow{x} T_pG \xrightarrow{\lambda_G} T_pG^\vee \xrightarrow{y^\vee} T_p\overline{\mathcal{E}}^\vee \xrightarrow{\lambda_{\overline{\mathcal{E}}}^\vee} T_p\overline{\mathcal{E}}) \in \mathrm{End}_{O_F}(T_p\overline{\mathcal{E}}) \cong O_F.$$

One can check that L is represented by the hermitian matrix $\mathrm{Diag}((1)^{n-h}, (\pi)^h)$.

Following [LZ22a, §4], we consider two injective O_F-linear isometric homomorphisms

$$i_K : \mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G)_F \to L_F,$$

and

$$i_{\bar{k}} : \mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G)_F \to \mathbb{V}.$$

By [LZ22a, Lemma 4.4.1], we have

$$\mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G) = i_K^{-1}(L). \tag{3.1}$$

By the definition of special cycles, for any O_F-lattice $M \subset \mathbb{V}$, we have $z \in \mathcal{Z}(M)(O_K)$ if and only if $M \subset i_{\bar{k}}(\mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G))$. By (3.1), $z \in \mathcal{Z}(M)(O_K)$ if and only if

$$M \subset i_{\bar{k}}(i_K^{-1}(L)). \tag{3.2}$$

Now assume that $z \in \mathcal{Z}(L^\flat)(O_K)$ corresponds to an O_F-hermitian module G of signature $(1, n-1)$ over O_K. By (3.2), we have

$$L^\flat \subset i_{\bar{k}}(i_K^{-1}(L)).$$

Define $M^\flat \coloneqq L_F^\flat \cap i_{\bar{k}}(i_K^{-1}(L))$. By (3.2), we still have $z \in \mathcal{Z}(M^\flat)(O_K)$. Moreover, we have

$$M^\flat \xrightarrow{\sim} L \cap i_K(i_{\bar{k}}^{-1}(L_F^\flat)).$$

Set $\mathbb{W} = i_K(i_{\bar{k}}^{-1}(L_F^\flat))$, which has the same dimension as $L_F^\flat$.

Definition 3.1. Define $H(\mathbb{V})$ to be the collection of O_F-lattices $M^\flat \subset \mathbb{V}$ of rank $n-1$ such that

$$M^\flat \approx \mathrm{Diag}((1)^{n-h}, (\pi)^{h-2}, (\pi^a)) \text{ with } a \in \mathbb{Z}_{>0} \quad \text{or} \quad M^\flat \approx \mathrm{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a)) \text{ with } a \in \mathbb{Z}_{\geq 0}.$$

Definition 3.2. For any lattice $L'^\flat$ such that $L^\flat \subset L'^\flat \subset (L'^\flat)^\vee \subset L_F^\flat$, we define the primitive part $\mathcal{Z}(L'^\flat)^\circ$ of the special cycle $\mathcal{Z}(L'^\flat)$ inductively by setting

$$\mathcal{Z}(L'^\flat)^\circ \coloneqq \mathcal{Z}(L'^\flat) - \sum_{\substack{L'^\flat \subset L''^\flat \\ L''^\flat \subset (L''^\flat)^\vee \subset L_F^\flat}} \mathcal{Z}(L''^\flat)^\circ.$$

Moreover, we define $\mathcal{Y}(L'^\flat)^\circ$ similarly.

Theorem 3.3. *Let $L^\flat \subseteq \mathbb{V}_n$ be a hermitian O_F-lattice of rank $n-1$. Then*

$$\mathcal{Z}(L^\flat)_{\mathscr{H}} = \bigcup_{\substack{L^\flat \subseteq M^\flat, \\ M^\flat \in H(\mathbb{V})}} \mathcal{Z}(M^\flat)^\circ_{\mathscr{H}}.$$

Moreover, we have the following.

1. *If $M^\flat \approx \mathrm{Diag}((1)^{n-h}, (\pi)^{h-2}, (\pi^a))$ with $a \in \mathbb{Z}_{>0}$, then $\mathcal{Z}(M^\flat)_{\mathscr{H}} \cong \mathcal{Z}(M^\flat) \simeq \mathcal{Z}(x) \subset \mathcal{N}_2^{[0]}$ with $\mathrm{val}(x) = a-1$, which is a quasi-canonical lifting of degree $a-1$.*
2. *If $M^\flat \approx \mathrm{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a))$ with $a \in \mathbb{Z}_{\geq 0}$, then*

$$\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}} = \sum_{\substack{M^\flat \subset M_1 \oplus N_2 \subset \pi^{-1}M^\flat \\ N_2 \approx (\pi^{-1})^h}} \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ.$$

Here, each $\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(N_2)^\circ$ is a quasi-canonical lifting of degree a. The summation index has cardinality:

$$\begin{cases} q^{2h} & \text{if } a \geq 2, \\ q^{2h}\dfrac{1-(-q)^{-(h+1)}}{1-(-q)^{-1}} & \text{if } a = 1. \end{cases}$$

The rest of this section is devoted to the proof of Theorem 3.3. We give the proof at the end of this section after some preparations.

First, by the similar method as in the appendix of [HSY23], we can prove the following two lemmas.

Lemma 3.4. *Assume $L = L_1 \oplus L_2$ where $L_1 \approx (1)^{n-h}$ and $L_2 \approx (\pi)^h$. Let x be a primitive vector in L.*

1. *If $\mathrm{Pr}_{L_1}(x)$ is primitive in L_1, then there exists $L_1' \approx (1)^{n-h}$ such that $x \in L_1'$.*
2. *If $\mathrm{Pr}_{L_1}(x)$ is not primitive in L_1, then there exists $L_1' \approx (1)^{n-h}$ and $L_2' \approx (\pi)^h$ such that $L = L_1' \oplus L_2'$ and $x \in L_2'$.*

Lemma 3.5. *Assume $L \approx (1)^{n-h}$ or $(\pi)^h$. Then for any primitive vectors $x, x' \in L$ with $q(x) = q(x')$, there exists $g \in \mathrm{U}(L)$ such that $g(x) = x'$.*

With the help of the above two lemmas, we can prove the following.

Lemma 3.6. *Assume L is a hermitian lattice represented by $\mathrm{Diag}((1)^{n-h}, (\pi)^h)$ and $\mathbb{W} \subset L_F$ is a subspace of dimension $n-1$. Then $M^\flat := \mathbb{W} \cap L$ is represented either by $\mathrm{Diag}(1^{n-h-2}, (\pi)^h, (\pi^a))$ with $a \in \mathbb{Z}_{\geq 0}$ or $\mathrm{Diag}((1)^{n-h}, (\pi)^{h-2}, (\pi^a))$ with $a \in \mathbb{Z}_{>0}$. In the first case, we can write $M^\flat = M_1 \oplus M_2$ such that $L = L_1 \oplus M_2$.*

Proof. First, we assume $M_1^\flat := M^\flat \cap L_{1,F}$ is not unimodular. Then the rank of L_1 is at least 2 and we choose a basis $\{e_1, \dots, e_{n-h}\}$ of L_1 such that $e_1, e_2 \in L_1$ and $(e_1, e_1) = (e_2, e_2) = 0$, $(e_1, e_2) = 1$ and $(e_i, e_j) = 0$ for $i, j > 2, i \neq j$. Since $M_1^\flat$ is primitive in L_1 but non-isometric to L_1, we know the rank of $M_1^\flat$ is smaller than $n-h$. Hence the rank of $M_1^\flat$ has to be $n-h-1$ and $M_2^\flat := M^\flat \cap L_{2,F} = L_2$. Now we choose a orthogonal basis $\{x_1, \dots, x_{n-h-1}\}$ of $M_1^\flat$, where we assume $q(x_1) = (x_1, x_1)$ has largest valuation among $\{q(x_1), \dots, q(x_{n-h-1})\}$. In particular, $\mathrm{val}(q(x_1)) > 0$. By Lemmas 3.4 and 3.5, we may assume $x_1 = e_1 + \frac{q(x_1)}{2} e_2$. Since $M_1^\flat$ is primitive, we can write $x_i = a_i(e_1 - \frac{q(x_1)}{2} e_2) + \sum_{j=3}^{n-h} a_{ij} e_j$ and for each j with $3 \leq j \leq n-h$, there exists an i such that a_{ij} is a unit. Hence by possibly choosing a different basis $\{x_2, \dots, x_{n-h-1}\}$, we may assume $x_i = a_i(e_1 - \frac{q(x_1)}{2})e_2) + e_{i+1}$. Now it is clear that $\mathrm{Span}\{x_2, \dots, x_{n-h-1}\}$ is unimodular and $M^\flat \approx \mathrm{Diag}((1)^{n-h-2}, (\pi)^h, (\pi^a))$.

Now we assume $M_1^\flat := M^\flat \cap L_{1,F}$ is unimodular. If $\mathrm{rank} M_1^\flat = n-h-1$, then an argument as above shows that $M^\flat \approx \mathrm{Diag}((1)^{n-h-1}, (\pi)^h)$. If If $\mathrm{rank} M_1^\flat = n-h$, then $\mathrm{rank} M_2^\flat = h-1$, and a similar argument as before shows that $M_2^\flat \approx (\pi)^{h-2} \oplus (\pi^a)$ with $a \in \mathbb{Z}_{>0}$. Then $M^\flat \approx (1)^{n-h} \oplus (\pi)^{h-2} \oplus (\pi^a)$ and $(1)^{n-h-1} \oplus (\pi)^h$ respectively. □

In particular, Lemma 3.6 implies that if $z \in \mathcal{Z}(L^\flat)(O_K)$, then $z \in \mathcal{Z}(M^\flat)(O_K)$ where $M^\flat \in H(\mathbb{V})$.

Now we assume $z \in \mathcal{Z}(L^\flat)(O_K)$ corresponds to a $M^\flat$ represented by $(1)^{n-h-2} \oplus (\pi^a) \oplus (\pi)^h$. By Proposition 2.6, we may reduce to the case $M^\flat$ represented by $(\pi^a) \oplus (\pi)^h$.

Proposition 3.7. *Assume $z \in \mathcal{Z}(L^\flat)(O_K)$ corresponds to a $M^\flat = M_1 \oplus M_2$, where M_1 is represented by (π^a) and M_2 is represented by $(\pi)^h$. If $L = L_1 \oplus M_2$, then $z \in \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1} M_2)^\circ$.*

Proof. First, notice that $z \in \mathcal{Z}(M^\flat)^\circ$ essentially by the definition of $M^\flat$. We may choose a basis $\{x_1, x_2\}$ of L_1 and a basis $\{x_3, \cdots, x_{h+2}\}$ of M_2 such that the moment matrix of $\{x_1, x_2\}$ and $\{x_3, \cdots, x_{h+2}\}$ is $(1)^2$ and $(\pi)^h$. Let

$$L_{z^\vee} := \mathrm{Hom}_{O_F}(T_p \overline{\mathcal{E}}, T_p G^\vee).$$

Composing L with λ_G, we obtain an embedding $L \hookrightarrow L_{z^\vee}$ such that $L_{z^\vee} = \lambda_G(L_1) \oplus \pi^{-1} \lambda_G(M_2)$. Moreover, the moment matrix of $\lambda_G(L_1) \oplus \pi^{-1} \lambda_G(M_2)$ is $(\pi)^2 \oplus (1)^h$. We can also directly check that $L_{z^\vee} \cap \lambda_G(i_K(i_{\bar{k}}^{-1}(L_F^\flat))) = M_1 \oplus \pi^{-1} \lambda_G(M_2)$. Hence, $z \in \mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1} M_2)^\circ$ by the definition of special cycles. □

Lemma 3.8. *Assume $M^\flat = M_1 \oplus M_2 = M_1' \oplus M_2$. Then*

$$\mathcal{Z}(M_1)^\circ \cdot \mathcal{Y}(\pi^{-1} M_2)^\circ = \mathcal{Z}(M_1')^\circ \cdot \mathcal{Y}(\pi^{-1} M_2)^\circ.$$

Proof. First of all, we have $\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(\pi^{-1}M_2)^\circ\subset\mathcal{Z}(M_1)^\circ\cdot\mathcal{Z}(M_2)^\circ=\mathcal{Z}(M_1')^\circ\cdot\mathcal{Z}(M_2)^\circ\subset\mathcal{Z}(M_1')^\circ$. Hence, $\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(\pi^{-1}M_2)^\circ\subset\mathcal{Z}(M_1')^\circ\cdot\mathcal{Y}(\pi^{-1}M_2)^\circ$. By switching the role of M_1 and M_1', the lemma is proved. □

Lemma 3.9. *Let $M^\flat=M_1\oplus M_2$, where $M_1=\mathrm{span}\{x_0\}$, and M_2 has a basis $\{x_1,\cdots,x_h\}$ with moment matrix $(\pi)^h$. Assume $M^\flat\subset N^\flat\subset\pi^{-1}M^\flat$ and $N^\flat\approx\mathrm{Diag}(\pi^a,(\pi^{-1})^h)$. Then $N^\flat=M_1\oplus\mathrm{span}\{\pi^{-1}(x_1+\alpha_1x_0),\cdots,\pi^{-1}(x_h+\alpha_hx_0)\}$, where α_i is a representative of $O_F/(\pi)$.*

Proof. Since $M^\flat\approx\mathrm{Diag}(\pi^a,(\pi)^h)$ and $N^\flat\approx\mathrm{Diag}(\pi^a,(\pi^{-1})^h)$, $N^\flat$ must contain a sub-lattice of the form $\mathrm{span}\{\pi^{-1}(x_1+\alpha_1x_0),\cdots,\pi^{-1}(x_h+\alpha_hx_0)\}$. Moreover, we have $M^\flat\overset{h}{\subset}N^\flat\subset\pi^{-1}M^\flat$.

Notice that $M^\flat\overset{h}{\subset}M_1\oplus\mathrm{span}\{\pi^{-1}(x_1+\alpha_1x_0),\cdots,\pi^{-1}(x_h+\alpha_hx_0)\}\subset N^\flat$. Hence $N^\flat=M_1\oplus\mathrm{span}\{\pi^{-1}(x_1+\alpha_1x_0),\cdots,\pi^{-1}(x_h+\alpha_hx_0)\}$. □

Lemma 3.10. *Assume $M^\flat=M_1\oplus M_2$ with basis $\{x_0,x_1,\cdots,x_h\}$ as in Lemma 3.9. Let $N^\flat$ and $(N^\flat)'$ be two different lattices such that $M^\flat\subset N^\flat\subset\pi^{-1}M^\flat$, $M^\flat\subset(N^\flat)'\subset\pi^{-1}M^\flat$, and $N^\flat\approx(N^\flat)'\approx\mathrm{Diag}(\pi^a,(\pi^{-1})^h)$. Write $N^\flat=M_1\oplus N_2$ and $(N^\flat)'=M_1\oplus N_2'$. Then*

$$\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(N_2)^\circ(O_K)\neq\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}^\circ(N_2')(O_K).$$

Proof. Let $N^\flat=M_1\oplus\mathrm{span}\{\pi^{-1}(x_1+\alpha_1x_0),\cdots,\pi^{-1}(x_h+\alpha_hx_0)\}$ and $(N^\flat)'=M_1\oplus\mathrm{span}\{\pi^{-1}(x_1+\alpha_1'x_0),\cdots,\pi^{-1}(x_h+\alpha_h'x_0)\}$ where α_i and α_i' are representatives of $O_F/(\pi)$. If $\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(N_2)^\circ(O_K)=\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}^\circ(N_2')(O_K)$, then we have nontrivial $z\in\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(N_2)^\circ(O_K)\cap\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}^\circ(N_2')(O_K)$. This implies that $z\in\mathcal{Y}(\pi^{-1}M_1)$. In particular, $z\in\mathcal{Y}(\pi^{-1}M_1)\cdot\mathcal{Y}(\mathrm{span}\{\pi^{-1}x_1,\cdots,\pi^{-1}x_h\})(O_K)$. However, notice that

$$\mathcal{Y}(\pi^{-1}M_1)\cdot\mathcal{Y}(\mathrm{span}\{\pi^{-1}x_1,\cdots,\pi^{-1}x_h\})\cong\mathcal{Z}(\pi^{-1}M_1)\cdot\mathcal{Y}(\mathrm{span}\{\pi^{-1}x_1,\cdots,\pi^{-1}x_h\})$$

by cancellation law and the fact that $\mathcal{Z}(\pi^{-1}M_1)=\mathcal{Y}(\pi^{-1}M_1)$ in $\mathcal{N}_2$. This contradicts the fact $z\notin\mathcal{Z}(\pi^{-1}M_1)^\circ$. □

Proof of Theorem 3.3. Assume $z\in\mathcal{Z}(M^\flat)^\circ(O_K)$. By Lemma 3.6, we know that $z\in\mathcal{Z}(M^\flat)(O_K)$ where $M^\flat\in H(\mathbb{V})$.

If $M^\flat=M_1\oplus M_2$, where $M_1\approx(1)^{n-h}$ and $M_2\approx\mathrm{Diag}((\pi)^{h-2},(\pi^a))$ with $a>0$, then according to Proposition 2.6, $\mathcal{Z}(M^\flat)\cong\mathcal{Z}(M_2)\subset\mathcal{N}_h^{[h]}$. Then applying the duality between $\mathcal{N}_h^{[h]}$ and $\mathcal{N}_h^{[0]}$, we have $\mathcal{Z}(M_2)\subset\mathcal{N}_h^{[h]}$ is isomorphic to $\mathcal{Z}(M_2')\subset\mathcal{N}_h^{[0]}$, where $M_2'\cong\mathrm{Diag}((1)^{h-2},(\pi^{a-1}))$. Then by Proposition 2.6 and [KR11, Proposition 8.1], $\mathcal{Z}(M^\flat)^\circ$ is isomorphic to a quasi-canonical lifting of degree $a-1$.

Now we assume $M^\flat\approx\mathrm{Diag}(1^{n-h-2},(\pi)^h,(\pi^a))$ with $a\in\mathbb{Z}_{\geq0}$. Then according to Proposition 3.7, and Lemma 3.9, we have $z\in\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(N_2)^\circ(O_K)$ for some N_2, where $N_2=\mathrm{span}\{\pi^{-1}(x_1+\alpha_1x_0),\cdots,\pi^{-1}(x_h+\alpha_hx_0)\}$ and each α_i is some representative of $O_F/(\pi)$. If $M_1\cong\mathrm{Diag}(1^{n-h-2},\pi^a)$ and $M_2\cong(\pi)^h$, then by Proposition 2.6 and [KR11, Proposition 8.1] again, $\mathcal{Z}(M_1)^\circ\cdot\mathcal{Y}(\pi^{-1}M_2)^\circ$ is isomorphic to a quasi-canonical lifting of degree a.

According to the above discussion, we have

$$\mathcal{Z}(L^\flat)_{\mathscr{H}}=\bigcup_{\substack{L^\flat\subseteq M^\flat,\\ M^\flat\in H(\mathbb{V})}}m(M^\flat)\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}},\tag{3.3}$$

and $\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}}$ has the desired description.

Now we show $m(M^\flat)=1$ for any $M^\flat$ in the above identity following the proof of [LZ22a, Theorem 4.2.1] closely. It suffices to show $O_K[\epsilon]$-points of both sides of (3.3) are the same where $\epsilon^2=0$. First, each O_K-point of the right hand side of (3.3) has a unique lift to an $O_K[\epsilon]$-point. Therefore we only need to show each $z\in\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}}(O_K)$ has a unique lift in $\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}}(O_K[\epsilon])$.

Let G be the corresponding O_F-hermitian module of signature $(1,n-1)$ over O_K and $\mathbb{D}(G)$ be the (covariant) O_{F_0}-relative Dieudonné crystal of G. First, we have an action $O_F\otimes_{O_{F_0}}O_K\simeq O_K\oplus O_K$ on $\mathbb{D}(G)(O_K)$ induced by the action of O_F via $\iota:O_F\to\mathrm{End}(G)$, and hence a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{D}(G)(O_K)$. Then we let $\mathscr{A}=\mathrm{gr}_0\,\mathbb{D}(G)(O_K)$. It is a free O_K-module of rank n equipped with an O_K-hyperplane: $\mathscr{H}:=\mathrm{Fil}^1\,\mathbb{D}(G)(O_K)\cap\mathscr{A}$ by the Kottwitz signature condition. Note that $\mathscr{H}$ contains the image of $L^\flat$ under the identification of [KR11, Lemma 3.9].

Note that the kernel of $O_K[\epsilon]\to O_K$ has a natural nilpotent divided power structure. Then according to Grothendieck-Messing theory, a lift $\tilde{z}\in\mathcal{Z}(L^\flat)(O_K[\epsilon])$ of z corresponds to an $O_K[\epsilon]$ direct summand of $\mathbb{D}(G)(O_K[\epsilon])$ $\widetilde{\mathrm{Fil}}$ that lifts $\mathrm{Fil}^1\mathbb{D}(G)$ and contains the image of $L^\flat$ in $\widetilde{\mathscr{A}}$. Here, $\widetilde{\mathrm{Fil}}$ is isotropic under the natural pairing $\langle\,,\,\rangle_{\mathbb{D}(G)(O_K[\epsilon])}$ on $\mathbb{D}(G)(O_K[\epsilon])$ induced by the polarization. Since $L^\flat\subset\mathrm{Hom}_{O_F}(\overline{\mathcal{E}},G)$ has rank $n-1$, by Breuil's theorem [LZ22a, §4.3], we know that the image of $L^\flat$ in $\mathrm{gr}_0\,\mathbb{D}(G)(S)$ has rank $n-1$ over S (the Breuil's ring) and thus its image in the base change $\mathscr{A}$ has rank $n-1$ over O_K. In particular, $\mathrm{gr}_0\widetilde{\mathrm{Fil}}$ is the unique $O_K[\epsilon]$-hyperplane $\widetilde{\mathscr{H}}$ of $\mathrm{gr}_0\mathbb{D}(G)(O_K[\epsilon])$ that contains the $O_K[\epsilon]$-module spanned by the image of $L^\flat$ in $\widetilde{\mathscr{A}}$.

To determine $\mathrm{gr}_1\widetilde{\mathrm{Fil}}$, we note that $\widetilde{\mathrm{Fil}}$ is a direct summand of $\mathbb{D}(G)(O_K[\epsilon])$ with rank n containing $\widetilde{\mathscr{H}}$. Since $\widetilde{\mathrm{Fil}}$ is isotropic under $\langle\,,\,\rangle_{\mathbb{D}(G)(O_K[\epsilon])}$, we have $\mathrm{gr}_1\widetilde{\mathrm{Fil}}\subset(\widetilde{\mathscr{H}})^\perp\cap\mathrm{gr}_1\mathbb{D}(G)(O_K[\epsilon])$. Here $(\widetilde{\mathscr{H}})^\perp$ is the perpendicular subspace in $\mathbb{D}(G)(O_K[\epsilon])$ with respect to $\langle\,,\,\rangle_{\mathbb{D}(G)(O_K[\epsilon])}$. Moreover, since $\det\langle\,,\,\rangle_{\mathbb{D}(G)(O_K[\epsilon])}\neq0$ in $O_K[\epsilon]$, we have $(\widetilde{\mathscr{H}})^\perp$ has rank $n+1$. Note that $\mathrm{gr}_0\mathbb{D}(G)(O_K[\epsilon])$ is also isotropic under $\langle\,,\,\rangle_{\mathbb{D}(G)(O_K[\epsilon])}$. In particular, $\mathrm{gr}_0\mathbb{D}(G)(O_K[\epsilon])\subset(\widetilde{\mathscr{H}})^\perp$ which has rank n. Hence $(\widetilde{\mathscr{H}})^\perp\cap\mathrm{gr}_1\mathbb{D}(G)(O_K[\epsilon])$ is of rank one. Since $\widetilde{\mathrm{Fil}}$ is a direct summand of $\mathbb{D}(G)(O_K[\epsilon])$, we know $\mathrm{gr}_1\widetilde{\mathrm{Fil}}=(\widetilde{\mathscr{H}})^\perp\cap\mathrm{gr}_1\mathbb{D}(G)(O_K[\epsilon])$. Hence $\widetilde{\mathrm{Fil}}=\mathrm{gr}_0\widetilde{\mathrm{Fil}}\oplus\mathrm{gr}_1\widetilde{\mathrm{Fil}}$ is uniquely determined, and the lift $\tilde{z}\in\mathcal{Z}(L^\flat)(O_K[\epsilon])$ of z is unique. Hence $\widetilde{\mathrm{Fil}}=\mathrm{gr}_0\widetilde{\mathrm{Fil}}\oplus\mathrm{gr}_1\widetilde{\mathrm{Fil}}$ is uniquely determined, and the lift $\tilde{z}\in\mathcal{Z}(L^\flat)(O_K[\epsilon])$ of z is unique.

We defer the proof of the cardinality of the summation index to Lemma 8.22, where it is proved via analytic method. □

4. Local modularity and Tate conjectures

First, as we have discussed in the introduction, we propose the following local modularity conjecture motivated by the analytic computation and the special case for $\mathcal{N}_n^{[0]}$ (see [LZ22a, Corollary 5.3.3]).

Conjecture 4.1. *For the Rapoport-Zink space $\mathcal{N}_n^{[h]}$ and an O_F-lattice $L^\flat\subset\mathbb{V}$ of rank $n-1$, we have*

$$\widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}}(\mathcal{Z}(x))=-\frac{1}{q^h}\mathrm{Int}_{L^\flat,\mathscr{V}}(\mathcal{Y}(x)).$$

Remark 4.2. Conjecture 4.1 is primarily motivated by Theorem 9.2 below along with [LZ22a, Lemma 6.3.1] and [Zha22, Theorem 8.1].

Conjecture 4.3. *(cf.* [LZ22a, Corollary 5.3.3]*) For the Rapoport-Zink space $\mathcal{N}_n^{[h]}$ and an O_F-lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$, there are finitely many Deligne-Lusztig curves $C_i \subset \mathcal{N}_3^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, projective lines $D_i \subset \mathcal{N}_2^{[1]} \hookrightarrow \mathcal{N}_n^{[h]}$ and $mult_{C_i}$, $mult_{D_i} \in \mathbb{Q}$ such that for any $x \in \mathbb{V}\backslash L_F^\flat$,*

$$\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_\mathscr{V} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) = \sum_i mult_{C_i}\chi(\mathcal{N}_n^{[h]}, O_{C_i} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) + \sum_i mult_{D_i}\chi(\mathcal{N}_n^{[h]}, O_{D_i} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}).$$

Remark 4.4. Conjecture 4.1 follows from Conjecture 4.3. Conjecture 4.3 is implied by a stronger version of Tate conjectures on 1-cycles for Deligne–Lusztig varieties Y_Λ in Proposition 4.6 (cf. [LZ22a, Theorem 5.3.2]).

Theorem 4.5. *Conjecture 4.3 holds for $\mathcal{N}_n^{[0]}, \mathcal{N}_n^{[1]}, \mathcal{N}_n^{[n-1]}, \mathcal{N}_n^{[n]}$, and $\mathcal{N}_4^{[2]}$.*

To prove this theorem, we follow [LZ22a, Section 5.3]. First, we need to recall several notations and theorems from [Cho18]. In [Cho18, Theorem 1.1], we proved that the reduced subscheme of $\mathcal{N}_n^{[h]}$ has a Bruhat-Tits stratification and their components are certain Deligne-Lusztig varieties Y_Λ where Λ is a vertex lattice of type $t(\Lambda)$. More precisely, we have the following proposition.

Proposition 4.6. [Cho18, Theorem 1.1] *Let $\mathcal{N}_{n,red}^{[h]}$ be the underlying reduced subscheme of $\mathcal{N}_n^{[h]}$. Then, we have*

$$\mathcal{N}_{n,red}^{[h]} = \cup_{t(\Lambda)\leq h-1} Y_\Lambda \cup \cup_{t(\Lambda)\geq h+1} Y_\Lambda,$$

where Y_Λ denotes certain Deligne-Lusztig varieties associated with vertex lattices Λ. Also, the dimension of Y_Λ is $\frac{1}{2}(t(\Lambda)+h-1)$ (resp. $\frac{1}{2}(t(\Lambda)+n-h-1)$) if $t(\Lambda) \geq h+1$ (resp. $t(\Lambda) \leq h-1$).

By [LZ22a, Corollary 5.3.3], we know that the Conjecture 4.3 holds for $\mathcal{N}_n^{[0]}$ and $\mathcal{N}_n^{[n]}$. Also, by [Cho18, Theorem 1.1], the irreducible components of the reduced subscheme of $\mathcal{N}_4^{[2]}$ are $\mathbb{P}^2$ and their Chow groups are well-known. Therefore, let us focus on $\mathcal{N}_n^{[1]}, \mathcal{N}_n^{[n-1]}$. Since $\mathcal{N}_n^{[1]}$ is isomorphic to $\mathcal{N}_n^{[n-1]}$, we only need to consider $\mathcal{N}_n^{[1]}$. In this case, by Proposition 4.6 (see [Cho18, Theorem 1.1] for more detail), we know that the reduced subscheme of $\mathcal{N}_n^{[1]}$ has a Bruhat-Tits stratification and their components are Deligne-Lusztig varieties Y_Λ where $t(\Lambda) \geq 2$ or projective spaces $\mathbb{P}^n_\Lambda$, where $t(\Lambda) = 0$. Let us describe Y_Λ more precisely.

Let k_F be the residue field of F and let V_{2d+h+1} be the unique (up to isomorphism) k_F/k-hermitian space of dimension $2d+h+1$. Let $\Lambda/\pi\Lambda^\vee = V_{2d+h+1}$ where $t(\Lambda) = 2d+h+1$, and let J_{2d+h+1} be the special unitary group associated to $(V_{2d+h+1}, (\cdot,\cdot))$. Let (W_{2d+h+1}, S_{2d+h+1}) be the Weyl system of J_{2d+h+1} and let B_{2d+h+1} be the standard Borel subgroup. Let $\mathcal{F} : J_{2d+h+1} \to J_{2d+h+1}$ be the Frobenius morphism over $k = \mathbb{F}_q$. For $I \subset S_{2d+h+1}$, we define W_I as the subgroup of W_{2d+h+1} generated by I and $P_I \coloneqq B_{2d+h+1}W_I B_{2d+h+1}$. Note that W_{2d+h+1} can be identified with a symmetric group S_{2d+h+1} with $\{s_1, \ldots, s_{2d+h}\}$ where s_i is the transposition of i and $i+1$. We write

$$I_0 \coloneqq \begin{cases} \{s_1, \ldots s_d, s_{d+2}, \ldots, s_{2d+1}\}, & \text{if } h = 1 \\ \{s_1, \ldots s_d, s_{d+2}, \ldots, s_{d+h}, s_{d+h+2} \ldots, s_{2d+h}\}, & \text{if } h > 1 \end{cases},$$

$$I_i \coloneqq \begin{cases} \{s_1, \ldots, s_{d-i}, s_{d+i+2}, \ldots, s_{2d+1}\}, 1 \leq i \leq d, & \text{if } h = 1 \\ \{s_1, \ldots s_{d-i}, s_{d+2}, \ldots, s_{d+h}, s_{d+h+i+2} \ldots, s_{2d+h}\}, 1 \leq i \leq d & \text{if } h > 1 \end{cases},$$

$$P_i \coloneqq P_{I_i},$$

$$w_\Lambda \coloneqq s_{d+1}s_{d+2}\ldots s_{d+h}.$$

Also, for $I \subset S_{2d+h+1}$ and $w \in W_{2d+h+1}$, we define the Deligne-Lusztig variety $X_I(w)$ by

$$X_I(w) := \{g \in J_{2d+h+1}/P_I | g^{-1}\mathcal{F}(g) \in P_I w P_{\mathcal{F}(I)}\}.$$

Finally, we define Y_Λ by $Y_\Lambda := \overline{X_{I_0}(w_\Lambda)} = X_{I_0}(id) \sqcup X_{I_0}(w_\Lambda)$ (see [Cho18, Definition 3.10]). Then, we have the following proposition.

Proposition 4.7. *Y_Λ is smooth.*

Proof. Note that Y_Λ is smoothly equivalent to the closure of $P_0 w_\Lambda \mathcal{F}(P_0)/\mathcal{F}(P_0) \subset J_{2d+h+1}/\mathcal{F}(P_0)$, a Schubert variety in the partial flag variety $J_{2d+h+1}/\mathcal{F}(P_0)$ ([GHN24, Section 7.1]). Also, this is smoothly equivalent to its inverse image in J_{2d+h+1}/B_{2d+h+1}, the Schubert variety for the maximal element in $P_0 w_\Lambda \mathcal{F}(P_0)$ ([GHN24, Section 7.2]). Now, by [LS90, Theorem 1] or [BL00, Theorem 8.1.1], it suffices to show that the maximal element in $P_0 w_\Lambda \mathcal{F}(P_0)$ avoids the patterns 3412, 4231. By the balls-in-boxes picture for w_Λ ([BKP+18, Section 3.2]), we can compute that the maximal element in $P_0 w_\Lambda \mathcal{F}(P_0)$ is

$$\begin{pmatrix} 1 & 2 & \dots & d & d+1 & \dots & d+h & d+h+1 & d+h+2 & \dots & 2d+h & 2d+h+1 \\ d+1 & d & \dots & 2 & d+h+1 & \dots & d+2 & 2d+h+1 & 2d+h & \dots & d+h+2 & 1 \end{pmatrix}.$$

It is now easy to see that this element avoids the patterns 3412, 4231, and hence Y_Λ is smooth. □

Now, assume that $h = 1$. In this case, we have that $P_d = B_{2d+2}$. Also, note that the elements in J_{2d+2}/P_i parametrize flags

$$0 \subset T_{-i} \overset{1}{\subset} T_{-i+1} \dots \overset{1}{\subset} T_{-1} \overset{1}{\subset} \overline{A} \overset{1}{\subset} \overline{B} \overset{1}{\subset} T_1 \dots \overset{1}{\subset} T_i \subset V_{2d+2}.$$

Now, by [Cho18, Theorem 1.1], we know that the reduced subscheme of $\mathcal{N}_n^{[1]}$ is

$$\mathcal{N}_{n,red}^{[1]} = \cup_{\Lambda, t(\Lambda) \geq 2} Y_\Lambda \cup \cup_{\Lambda, t(\Lambda)=0} \mathbb{P}^n_\Lambda. \tag{4.1}$$

Here, $Y_\Lambda = X_{P_0}(id) \sqcup X_{P_0}(s_{d+1})$ where $2d+2 = t(\Lambda)$(see [Cho18, Definition 3.10]). By [Cho18, Lemma 2.21] (cf. [Vol10, Lemma 2.1]), we have the following statement (cf. [Vol10, Theorem 2.15]).

Proposition 4.8. *(cf. [Vol10, Theorem 2.15]) There is a decomposition of $X_{P_0}(id) \sqcup X_{P_0}(s_{d+1})$ into a disjoint union of locally closed subvarieties*

$$X_{P_0}(id) \sqcup X_{P_0}(s_{d+1}) = \sqcup_{i=0}^{d} \{X_{P_i}(s_{d+2} \dots s_{d+i+1}) \sqcup X_{P_i}(s_{d+1} s_{d+2} \dots s_{d+i+1})\}.$$

Proof. We can follow the proof of [Vol10, Theorem 2.15] with [Cho18, Lemma 2.21]. □

Since $P_d = B_{2d+2}$, we have that $X_{P_d}(s_{d+2} \dots s_{2d+1})$ and $X_{P_d}(s_{d+1} s_{d+2} \dots s_{2d+1})$ are classical Deligne-Lusztig varieties. For $0 \leq i \leq d$, let us write

$$\begin{aligned}
&X^\circ_{i,1} := X_{P_i}(s_{d+1} \dots s_{d+i+1}), \\
&X^\circ_{i-1,2} := X_{P_i}(s_{d+2} \dots s_{d+i+1}), \\
&Y^\circ_{i,1} := X_{B_{2i+2}}(s_{i+1} \dots s_{2i+1}), \\
&Y^\circ_{i-1,2} := X_{B_{2i+2}}(s_{i+2} \dots s_{2i+1}), \\
&\widetilde{X}^\circ_i := X^\circ_{i,1} \sqcup X^\circ_{i,2}, \\
&\widetilde{X}_i := \sqcup_{m=0}^{i} \widetilde{X}^\circ_i, \\
&Y_d := \sqcup_{i=0}^{d} \{X_{P_i}(s_{d+2} \dots s_{d+i+1}) \sqcup X_{P_i}(s_{d+1} s_{d+2} \dots s_{d+i+1})\}.
\end{aligned}$$

Then, by the above proposition and a Bruhat-Tits stratification in [Cho18, Theorem 1.1], we have that $X^\circ_{i,1}$ (resp. $X^\circ_{i,2}$) is a disjoint union of isomorphic copies of $Y^\circ_{i,1}$ (resp. $Y^\circ_{i,2}$). Also, the dimensions of $Y^\circ_{i,1}$ and $Y^\circ_{i,2}$ are $i+1$.

For any k_F-variety S, we write $H^j(S)(i)$ for $H^j(S_{\overline{k}}, \overline{\mathbb{Q}_l}(i))$ where $l \neq p$ is a prime and $\overline{k}$ is an algebraically closed field containing k_F. Let $\mathscr{F} = \mathrm{Fr}_{k_F}$ be the q^2-Frobenius on $H^j(S)(i)$. Then, the following analogous statement of [LZ22a, Lemma 5.3.1] holds.

Lemma 4.9. *(cf. [LZ22a, Lemma 5.3.1]) For any $d, i \geq 0$ and $s \geq 1$, the action of $\mathscr{F}^s$ on the following cohomology groups are semisimple, and the space of $\mathscr{F}^s$-invariants is zero when $j \geq 1$.*

(1) $H^{2j}(Y^\circ_{d,1})(j)$.
(2) $H^{2j}(Y^\circ_{d-1,2})(j)$.
(3) $H^{2j}(\widetilde{X}^\circ_i)(j)$.
(4) $H^{2j}(Y_d - \widetilde{X}_i)(j)$.

Proof. Here, we follow the proof of [LZ22a, Lemma 5.3.1] with some modification.

(1) By [Lus76, (7.3) $({}^2\mathrm{A}_{2d+1}, \mathscr{F})$] (or we refer to the proof of [Ohm10, Lemma 2]), we have the following table on the eigenvalues of $\mathscr{F}$ on $H^j_c(Y^\circ_{d,1})$.

j	$d+1$	$d+2$	$d+3$	$\dots$	$2d-1$	$2d$	$2d+1$	$2d+2$
	1	q^2	q^4	$\dots$	q^{2d-4}	q^{2d-2}	q^{2d}	q^{2d+2}
	$-q^3$	$-q^5$	$-q^7$	$\dots$	$-q^{2d-1}$			.

By the Poincaré duality, we have a perfect pairing

$$H^{2d+2-j}_c(Y^\circ_{d,1}) \times H^j(Y^\circ_{d,1})(d+1) \to H^{2d+2}_c(Y^\circ_{d,1})(d+1) \simeq \overline{\mathbb{Q}}_l.$$

Therefore, the eigenvalues of $\mathscr{F}$ on $H^{2j}(Y^\circ_{d,1})(j)$ are given by $q^{2(d+1-j)}$ times the inverse of the eigenvalues in $H^{2(d+1-j)}_c(Y^\circ_{d,1})$. More precisely,

	$H^{2(d+1-j)}_c(Y^\circ_{d,1})$	$q^{2(d+1-j)} \times$ the inverse
$j=0$	q^{2d+2}	1
$j=1$	q^{2d-2}	q^2
$j \geq 2$	$q^{2d+2-4j}, -q^{2d-4j+5}$	$q^{2j}, -q^{2j-3}$.

Therefore, the eigenvalue of $\mathscr{F}^s$ cannot be 1 when $j \geq 1$. The semisimplicity of the action of $\mathscr{F}^s$ is from [Lus76, 6.1].

(2) By [Lus76, (7.3) $(\mathrm{A}_{d+1}, \mathscr{F})$] (or we refer to the proof of [Ohm10, Lemma 2], here, note that $\mathscr{F}$ is q^2-Frobenius), we have the following table on the eigenvalues of $\mathscr{F}$ on $H^j_c(Y^\circ_{d,2})$.

j	$d+1$	$d+2$	$d+3$	$\dots$	$2d-1$	$2d$	$2d+1$	$2d+2$
	1	q^2	q^4	$\dots$	q^{2d-4}	q^{2d-2}	q^{2d}	q^{2d+2}.

By the Poincaré duality, we have a perfect pairing

$$H^{2d+2-j}_c(Y^\circ_{d,2}) \times H^j(Y^\circ_{d,2})(d+1) \to H^{2d+2}_c(Y^\circ_{d,2})(d+1) \simeq \overline{\mathbb{Q}}_l.$$

Therefore, the eigenvalues of $\mathscr{F}$ on $H^{2j}(Y^\circ_{d,2})(j)$ are given by $q^{2(d+1-j)}$ times the inverse of the eigenvalues in $H_c^{2(d+1-j)}(Y^\circ_{d,2})$. More precisely,

	$H_c^{2(d+1-j)}(Y^\circ_{d,2})$	$q^{2(d+1-j)} \times$ the inverse
$j=0$	q^{2d+2}	1
$j=1$	q^{2d-2}	q^2
$j\geq 2$	$q^{2d+2-4j}$	q^{2j}.

Therefore, the eigenvalue of $\mathscr{F}^s$ cannot be 1 when $j \geq 1$. The semisimplicity of the action of $\mathscr{F}^s$ is from [Lus76, 6.1].

(3) This follows from (1) and (2) since $\widetilde{X}_i^\circ$ is a disjoint union of $Y^\circ_{i,1}$ and $Y^\circ_{i,2}$.

(4) This follows from (3) since $Y_d - \widetilde{X}_i^\circ = \sqcup_{m=i+1}^d \widetilde{X}_m^\circ$.

□

Theorem 4.10. *(cf. [LZ22a, Theorem 5.3.2]) For any $0 \leq i \leq d+1$ and any $s \geq 1$, we have*

(1) The space of Tate classes $H^{2i}(Y_d)(i)^{\mathscr{F}^s=1}$ is spanned by the cycle classes of the irreducible components of $\widetilde{X}_i$.

(2) Let $H^{2i}(Y_d)(i)_1 \subset H^{2i}(Y_d)(i)$ be the generalized eigenspace of $\mathscr{F}^s$ for the eigenvalue 1. Then $H^{2i}(Y_d)(i)_1 = H^{2i}(Y_d)(i)^{\mathscr{F}^s=1}$.

Proof. The proof is the same as [LZ22a, Theorem 5.3.2] with Lemma 4.9 □

Proof of Theorem 4.5. Here, we follow the proof of [LZ22a, Corollary 5.3.3]. For $\mathcal{N}_n^{[0]}$ and $\mathcal{N}_n^{[n]}$, this is from [LZ22a, Corollary 5.3.3]. For $\mathcal{N}_n^{[1]}$ and $\mathcal{N}_n^{[n-1]}$, note that $\mathcal{N}_n^{[1]}$ and $\mathcal{N}_n^{[n-1]}$ are isomorphic, so we only need to consider the case $\mathcal{N}_n^{[1]}$. By [Cho18, Theorem 1.1], we have (4.1):

$$\mathcal{N}_{n,red}^{[1]} = \cup_{\Lambda, t(\Lambda)\geq 2} Y_\Lambda \cup \cup_{\Lambda, t(\Lambda)=0} \mathbb{P}^n_\Lambda,$$

and any curve C in $\mathcal{N}_{n,red}^{[1]}$ lies in some $Y_\Lambda \simeq Y_d$ or the projective space $\mathbb{P}^n_\Lambda$ for some vertex lattice Λ. If it lies on Y_d, then by Theorem 4.10, the cycle class of C can be written as a $\mathbb{Q}$-linear combination of the cycle classes of the irreducible components of $\widetilde{X}_1$ and these are projective lines. Similarly, if it lies on $\mathbb{P}^n_\Lambda$, then by the Chow group of $\mathbb{P}^n_\Lambda$, we have that the cycle class of C can be written as a $\mathbb{Q}$-linear combination of projective lines. This finishes the proof of the theorem for $\mathcal{N}_n^{[1]}$ and $\mathcal{N}_n^{[n-1]}$.

For $\mathcal{N}_4^{[2]}$, we know that the irreducible components of $\mathcal{N}_{4,red}^{[2]}$ are $\mathbb{P}^2_\Lambda$ for vertex lattices Λ, and hence by the description of Chow group of $\mathbb{P}^2_\Lambda$, the cycle class of C can be written as a $\mathbb{Q}$-linear combination of projective lines. Moreover, the projective lines here can be realized as images of embeddings of projective lines in $\mathcal{N}_2^{[1]}$.

Finally, the finiteness of curves is from Lemma 2.7. This finishes the proof of the theorem. □

5. Weighted representation densities and conjectures

In this section, we first recall the definition of weighted representation densities and formulas from [Cho22b, Section 3.1]. Then, we will recall the conjectural formula in [Cho22b, Conjecture 3.17, Conjecture 3.25].

We denote by * the nontrivial Galois automorphism of F over F_0. We fix the standard additive character $\psi : F_0 \to \mathbb{C}^\times$ that is trivial on O_{F_0}. Let V^+ (resp. V^-) be a split (resp. non-split) $2n$-dimensional hermitian vector space over F and let $\mathcal{S}((V^\pm)^{2n})$ be the space of Schwartz functions on $(V^\pm)^{2n}$. Let $V_{r,r}$ be the split hermitian space of signature (r,r) and let $L_{r,r}$ be a self-dual lattice in $V_{r,r}$. Let $\phi_{r,r}$ be the characteristic function of $(L_{r,r})^{2n}$. Let $(V^\pm)^{[r]}$ be the space $V^\pm \otimes V_{r,r}$. For any function $\phi \in \mathcal{S}((V^\pm)^{2n})$, we define a function $\phi^{[r]}$ by $\phi \otimes \phi_{r,r} \in \mathcal{S}(((V^\pm)^{[r]})^{2n})$.

Let Γ_n be the Iwahori subgroup

$$\Gamma_n \coloneqq \{\gamma = (\gamma_{ij}) \in GL_n(O_F) \mid \gamma_{ij} \in \pi O_F \text{ if } i > j\}.$$

We define the set $V_n(F)$ and $X_n(F)$ as follows:

$$V_n(F) = \{Y \in M_{n,n}(F) \mid {}^tY^* = Y\},$$

$$X_n(F) = \{X \in GL_n(F) \mid {}^tX^* = X\}.$$

For $g \in GL_n(F)$ and $X \in X_n(F)$, we define the group action of $GL_n(F)$ on $X_n(F)$ by $g \cdot X = gX^tg^*$. For $X, Y \in V_n(F)$, we denote by $\langle X, Y\rangle = \mathrm{Tr}(XY)$. For $X \in M_{m,n}(F)$ and $A \in V_m(F)$, we denote by $A[X] =^t X^*AX$. For a hermitian matrix $A \in X_m(F)$, we define

$$A^{[r]} = \begin{pmatrix} A & \\ & I_{2r} \end{pmatrix}.$$

Now, let us recall the definition of usual representation densities.

Definition 5.1. For $A \in X_m(O_F)$ and $B \in X_n(O_F)$, we define $\mathrm{Den}(A, B)$ by

$$\mathrm{Den}(A,B) = \lim_{d\to\infty} (q^{-d})^{n(2m-n)} |\{x \in M_{m,n}(O_F/\pi^d O_F) \mid A[x] \equiv B (\mathrm{mod} \pi^d)\}|.$$

Now, let us recall the definition of weighted representation densities in [Cho22b].

Definition 5.2. [Cho22b, Definition 3.1] Let $0 \le h, t \le n$. Let L_t be a lattice of rank $2n$ in V^+ if t is even (resp. in V^- if t is odd) with hermitian form

$$A_t \coloneqq \begin{pmatrix} I_{2n-t} & \\ & \pi^{-1}I_t \end{pmatrix}.$$

Let $1_{h,t} \in \mathcal{S}((V^\pm)^{2n})$ be the characteristic function of $(L_t^\vee)^{2n-h} \times L_t^h$.

For $B \in X_{2n}(F)$, we define

$$W_{h,t}(B, (-q)^{-2r}) \coloneqq \int_{V_{2n}(F)} \int_{M_{2n+2r,2n}(F)} \psi(\langle Y, A_t^{[r]}[X] - B\rangle) 1_{h,t}^{[r]}(X) dX dY.$$

Here, dY (resp. dX) is the Haar measure on $V_{2n}(F)$ (resp. $M_{2n+2r,2n}(F)$) such that

$$\int_{V_{2n}(O_F)} dY = 1 \text{ (resp. } \int_{M_{2n+2r,2n}(O_F)} dX = 1).$$

The functions $\mathrm{Den}(A, B)$ and $W_{h,t}(B, r)$ have the following formulas.

Lemma 5.3. *([Hir00], [Cho22b, Lemma 3.5]) For $A \in X_m(F)$ and $B \in X_{2n}(F)$, we have that*

$$\mathrm{Den}(A^{[r]}, B) = \sum_{Y \in \Gamma_{2n} \backslash X_{2n}(F)} \frac{\mathcal{G}(Y,B)\mathcal{F}(Y, A^{[r]})}{\alpha(Y;\Gamma_{2n})},$$

$$W_{h,t}(B, (-q)^{-2r}) = \sum_{Y \in \Gamma_{2n} \backslash X_{2n}(F)} \frac{\mathcal{G}(Y,B)\mathcal{F}_h(Y, A_t^{[r]})}{\alpha(Y;\Gamma_{2n})}.$$

Here, we define $\mathcal{F}(Y, A^{[r]})$, $\mathcal{F}_h(Y, A_t^{[r]})$, and $\mathcal{G}(Y,B)$ as follows:

$$\mathcal{F}(Y, A^{[r]}) \coloneqq \int_{M_{m,2n}(F)} \psi(\langle Y, A^{[r]}[X]\rangle) dX,$$
$$\mathcal{F}_h(Y, A_t^{[r]}) \coloneqq \int_{M_{2n+2r,2n}(F)} \psi(\langle Y, A_t^{[r]}[X]\rangle) 1_{h,t}^{[r]}(X) dX,$$
$$\mathcal{G}(Y,B) \coloneqq \int_{\Gamma_{2n}} \psi(\langle Y, -B[\gamma]\rangle) d\gamma,$$

where $d\gamma$ is the Haar measure on $M_{2n,2n}(O_F)$ such that $\int_{M_{2n,2n}(O_F)} d\gamma = 1$.

Also, we define $\alpha(Y;\Gamma_{2n})$ by $\alpha(Y;\Gamma_{2n}) \coloneqq \lim_{d\to\infty} q^{-4dn^2} N_d(Y;\Gamma_{2n})$, where $N_d(Y;\Gamma_{2n}) = |\{\gamma \in \Gamma_{2n}(mod\ \pi^d) | \gamma \cdot Y \equiv Y (mod\ \pi^d)\}|$.

Definition 5.4. For $r \geq 0$, $A \in X_{n+2r}(O_F)$, and $B \in X_n(O_F)$, we can regard $\mathcal{F}(Y, A^{[r]})$, $\mathrm{Den}(A^{[r]}, B)$, $\mathcal{F}_h(Y, A_t^{[r]})$, and $W_{h,t}(B, (-q)^{-2r})$ as functions of $X = (-q)^{-2r}$. We define

$$\mathcal{F}'(Y, A) \coloneqq -\frac{d}{dX}\mathcal{F}(Y, A^{[r]})|_{X=1}, \text{ and } \mathcal{F}_h'(Y, A_t) \coloneqq -\frac{d}{dX}\mathcal{F}_h(Y, A_t^{[r]})|_{X=1}.$$

Also, we define

$$\mathrm{Den}'(A, B) = -\frac{d}{dX}\mathrm{Den}(A, B; X)|_{X=1}, \text{ and } W_{h,t}'(B) \coloneqq -\frac{d}{dX}W_{h,t}(B, r)|_{X=1}.$$

Definition 5.5. [Cho23, Proposition 2.7] For $0 \leq i, h \leq 2n$, we define the constant β_i^h by

$$\beta_i^h = \alpha_{i+1,h}^{-1} \left(\frac{\prod_{1 \leq m \leq 2n, m \neq i+1} (1 - x_m)}{\prod_{1 \leq m \leq 2n+1, m \neq i+1} (x_m - x_{i+1})} \right),$$

where

$$\begin{aligned}
&\alpha_{i,h} = (-q)^{(n+1-i)(2n-h)}, && x_i = (-q)^{n+1-i}, && 1 \leq i \leq n;\\
&\alpha_{i,h} = (-q)^{(2n+1-i)(2n+h)}, && x_i = (-q)^{i-2n-1}, && n+1 \leq i \leq 2n;\\
&\alpha_{2n+1,h} = 1, && x_{2n+1} = 1. &&
\end{aligned}$$

Now, we can state [Cho22b, Conjecture 3.17, Conjecture 3.25].

Conjecture 5.6. [Cho22b, Conjecture 3.17, Conjecture 3.25]

For a basis $\{x_1, \ldots, x_{2n-m}, y_1, \ldots, y_m\}$ of $\mathbb{V}$, and special cycles $\mathcal{Z}(x_1), \ldots, \mathcal{Z}(x_{2n-m})$, $\mathcal{Y}(y_1), \ldots, \mathcal{Y}(y_m)$ in $\mathcal{N}_{2n}^{[n]}$, we have

$$\chi(\mathcal{N}_{2n}^{[n]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{\mathcal{Y}(y_m)}) = \frac{1}{W_{n,n}(A_n, 1)} \{W_{m,n}'(B) - \sum_{0 \leq i \leq n-1} \beta_i^m W_{m,i}(B, 1)\}.$$

Here, χ is the Euler-Poincaré characteristic, $\otimes^{\mathbb{L}}$ is the derived tensor product, and B is the matrix

$$B = \begin{pmatrix} (x_i, x_j) & (x_i, y_l) \\ (y_k, x_j) & (y_k, y_l) \end{pmatrix}_{1 \le i,j \le 2n-m, 1 \le k,l \le m}.$$

Assume that special homomophisms $\{x_1, x_2, \dots, x_{n+h}, y_1, \dots, y_{n-h}\}$ has the hermitian matrix:

$$B = \begin{pmatrix} (x_i, x_j) & (x_i, y_l) \\ (y_k, x_j) & (y_k, y_l) \end{pmatrix}_{\substack{1 \le i,j \le n+h, \\ 1 \le k,l \le n-h}} = \begin{pmatrix} T & & \\ & I_h & \\ & & \pi^{-1} I_{n-h} \end{pmatrix}, \tag{5.1}$$

for some $n \times n$ matrix T. Then, by Proposition 2.6, the arithmetic intersection number of special cycles $\chi(\mathcal{N}_{2n}^{[n]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_{n+h})} \otimes^{\mathbb{L}} O_{\mathcal{Y}(y_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{\mathcal{Y}(y_{n-h})})$ in $\mathcal{N}_{2n}^{[n]}$ can be identified with $\mathrm{Int}_{n,h}(T) = \mathrm{Int}_{n,h}(L) = \chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_n)})$ in $\mathcal{N}_n^{[h]}$, where $L = \mathrm{Span}_{O_F}\{x_1, \cdots, x_n\}$. We note that the valuation of the determinant of B and $h+1$ have the same parity. Now, Conjecture 5.6 is specialized to the following conjecture.

Conjecture 5.7. *Consider a basis $\{x_1, \dots, x_{n+h}, y_1, \dots, y_{n-h}\}$ of $\mathbb{V}$ with moment matrix B as in (5.1). Let $L = \mathrm{Span}_{O_F}\{x_1, \cdots, x_n\}$. Then*

$$\mathrm{Int}_{n,h}(L) = \mathrm{Int}_{n,h}(T) = \frac{1}{W_{n,n}(A_n, 1)} \{W'_{n-h,n}(B) - \sum_{0 \le i \le n-1} \beta_i^{n-h} W_{n-h,i}(B, 1)\}. \tag{5.2}$$

Note that $\mathrm{Int}_{n,h}(L)$ is exactly the intersection number considered in Conjecture 1.3. We show the analytic sides of Conjectures 1.3 and 5.7 also match in §6 (see Corollary 6.7).

6. Cho-Yamauchi constants

In this section, we will modify the result of [Cho23] to get the Cho-Yamauchi constants in the case of $\mathcal{N}_n^{[h]}$. More precisely, we want to write the conjectural formula for the arithmetic intersection numbers of special cycles $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} O_{\mathcal{Z}(x_n)})$ in $\mathcal{N}_n^{[h]}$ as a linear sum of representation densities. First, we start with the following proposition.

Proposition 6.1. *Assume that B is of the form in (5.1). Then, we have*

$$\begin{aligned} W'_{n-h,n}(B) &= q^{-4n^2+(n+h)(n-h)} \mathrm{Den}(\pi A_n, I_{n-h}) \mathrm{Den}'(I_{n+h,h}, \begin{pmatrix} T & \\ & I_h \end{pmatrix}) \\ &= q^{-4n^2+(n+h)(n-h)} \mathrm{Den}(\pi A_n, I_{n-h}) \mathrm{Den}(I_{n+h,h}, I_h) \mathrm{Den}'(I_{n,h}, T). \end{aligned}$$

Proof. One can use a similar method as in [KR11, Corollary 9.12] to prove this. For example, see [Cho22b, Proposition A.3, Proposition A.4, (A.0.4), (A.0.5)]. □

Note that in Proposition 6.1, the terms $\mathrm{Den}(\pi A_n, I_{n-h})$ and $\mathrm{Den}(I_{n+h,h}, I_h)$ are constants, and $\mathrm{Den}'(I_{n,h}, T)$ is the derivative of a usual representation density. Now, by [Cho23], this can be written as a linear sum of usual representation densities. Let us follow the steps in [Cho23, Section 4.1]. For this, we need to introduce some notations.

Definition 6.2.

(1) We write $\mathcal{R}_n$ for the set $\mathcal{R}_n = \{Y_{\sigma,e} \mid (\sigma, e) \in \mathcal{S}_n \times \mathbb{Z}^n, \sigma^2 = 1, e_i = e_{\sigma(i)}, \forall i\}$, where $\mathcal{S}_n$ is the symmetric group of degree n, and $Y_{\sigma,e} = \sigma\mathrm{Diag}(\pi^{e_1}, \pi^{e_2}, \dots, \pi^{e_n})$. Then $\mathcal{R}_n$ forms a complete set of representatives of $\Gamma_n \backslash X_n(F)$.
(2) We write $\mathcal{R}_n^{0+}$ for the set $\mathcal{R}_n^{0+} = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$.
(3) For $\lambda \in \mathcal{R}_n^{0+}$, we define A_λ by $A_\lambda = \mathrm{Diag}(\pi^{\lambda_1}, \pi^{\lambda_2}, \dots \pi^{\lambda_n})$.
(4) For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{R}_n^{0+}$, we define $|\lambda|$ by $|\lambda| = \sum_{i=1}^n \lambda_i$.

Definition 6.3 (Cho-Yamauchi constant)**.** Assume that B is of the form in (5.1). For $\lambda \in \mathcal{R}_n^{0+}$, we define $D_{n,h}(\lambda)$ to be the constant satisfying

$$\frac{1}{W_{n,n}(A_n, 1)}\{W'_{n-h,n}(B) - \sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1)\} = \sum_{\lambda \in \mathcal{R}_n^{0+}} D_{n,h}(\lambda) \frac{\mathrm{Den}(A_\lambda, T)}{\mathrm{Den}(A_\lambda, A_\lambda)}.$$

The existence and uniqueness of these constants are from [Cho23]. The constant $D_{n,h}(\lambda)$ is a version of the Cho-Yamauchi constant in [CY20].

Remark 6.4. In $\mathcal{N}^h(1, n-1)$, let L be a rank n O_F-lattice generated by special homomorphisms $x_1, \dots, x_n$ in $\mathbb{V}$. Assume that T is the hermitian matrix of L. Then, the valuation of the determinant of T and $h+1$ have the same parity. Therefore, in Definition 6.3, the terms $\mathrm{Den}(A_\lambda, T)$ such that

$$\mathrm{val}(\det(A_\lambda)) = \sum_i \lambda_i \not\equiv h + 1 (\mathrm{mod}\ 2)$$

are always equal to 0.

Now, let us compute the correction terms $\dfrac{-\sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1)}{W_{n,n}(A_n, 1)}$.

Proposition 6.5. *Assume that B is of the form in (5.1). If $n - h \leq t \leq n-1$, we have*

$$W_{n-h,t}(B, 1) = q^{-4n^2 + (n+h)(3n-2t-h)} \mathrm{Den}(\pi A_t, I_{n-h}) \mathrm{Den}(I_{n+h,t-n+h}, I_h) \mathrm{Den}(I_{n,t-n+h}, T).$$

If $t < n - h$, we have that $W_{n-h,t}(B, 1) = 0$.

Proof. One can use a similar method as in [KR11, Corollary 9.12] to prove this. For example, see [Cho22b, Proposition A.3, Proposition A.4, (A.0.4), (A.0.5)]. □

Proposition 6.6. *For $n - h \leq t \leq n-1$, we have*

$$\frac{\beta_t^{n-h} W_{n-h,t}(B, 1)}{W_{n,n}(A_n, 1)} = \frac{-(-q)^{\frac{(n-t)(n-t-1-2h)}{2}}}{1 - (-q)^{-(n-t)}} \frac{\mathrm{Den}(I_{n,t-n+h}, T)}{\mathrm{Den}(I_{n,t-n+h}, I_{n,t-n+h})}.$$

Proof. This follows from Definition 5.5, [Cho22b, Proposition A.4], and Proposition 6.5. □

Combining Proposition 6.1 and Proposition 6.6, we get the following corollary which compares the analytic sides of Conjecture 1.3 and Conjecture 5.6.

Corollary 6.7. *Assume that B is of the in (5.1). By Proposition 6.1, Proposition 6.5, Proposition 6.6, and* [Cho22b, Proposition A.4]*, we can write* $\dfrac{1}{W_{n,n}(A_n, 1)}\{W'_{n-h,n}(B) - \sum_{0 \leq t \leq n-1} \beta_t^{n-h} W_{n-h,t}(B, 1)\}$

in terms of usual representation densities as follows:

$$\frac{1}{W_{n,n}(A_n,1)}\{W'_{n-h,n}(B)-\sum_{0\le t\le n-1}\beta_t^{n-h}W_{n-h,t}(B,1)\}$$
$$=\frac{\mathrm{Den}'(I_{n,h},T)}{\mathrm{Den}(I_{n,h},I_{n,h})}+\sum_{k=0}^{h-1}\frac{(-q)^{-\frac{(h-k)(h+k+1)}{2}}}{1-(-q)^{-(h-k)}}\frac{\mathrm{Den}(I_{n,k},T)}{\mathrm{Den}(I_{n,k},I_{n,k})}.$$

By Corollary 6.7, Conjecture 5.7 can be rewritten as follows.

Conjecture 6.8 ($\mathcal{Z}$-cycles in $\mathcal{N}_n^{[h]}$)**.** *For a basis $\{x_1,\dots,x_n\}$ of $\mathbb{V}$, and special cycles $\mathcal{Z}(x_1),\dots,\mathcal{Z}(x_n)$ in $\mathcal{N}_n^{[h]}$, we have*

$$\chi(\mathcal{N}_n^{[h]},O_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}O_{\mathcal{Z}(x_n)})=\frac{\mathrm{Den}'(I_{n,h},T)}{\mathrm{Den}(I_{n,h},I_{n,h})}+\sum_{k=0}^{h-1}\frac{(-q)^{-\frac{(h-k)(h+k+1)}{2}}}{1-(-q)^{-(h-k)}}\frac{\mathrm{Den}(I_{n,k},T)}{\mathrm{Den}(I_{n,k},I_{n,k})}.$$

Here, χ is the Euler-Poincaré characteristic and $\otimes^{\mathbb{L}}$ is the derived tensor product. Also, T is the matrix $T=\left((x_i,x_j)\right)_{1\le i,j\le n}$.

Proposition 6.9. *Conjecture 1.3 is equivalent to Conjecture 6.8.*

Proof. The intersection numbers from both conjectures are by definition the same. Hence we only need to show the analytic sides of both conjectures agree. By Corollary 6.7, we only need to show β_t^{n-h} from Conjecture 5.6 is the same as $c_{n,t}$ from Conjecture 1.3. Since the $c_{n,t}$ is characterized by $\partial\mathrm{Pden}_{n,h}(I_{n,t})=0$ for $t\le h-1$ and $t\equiv h+1 \pmod 2$, we only need to show $D_{n,h}(I_{n,t})=0$ for $t\le h-1$ and $t\equiv h+1 \pmod 2$. This is proved in Proposition 7.6 via a method we used throughout §7, so we postpone the proof. □

Remark 6.10. Before we start to find the constants $D_{n,h}(\lambda)$, let us provide a brief explanation of the forthcoming steps. By Proposition 6.1, we know that $\dfrac{W'_{n-h,n}(B)}{W_{n,n}(A_n,1)}$ is a constant multiple of $\mathrm{Den}'(I_{n,h},T)$. Also, we know how to write $\mathrm{Den}'(I_n,T)$ in terms of a linear sum of representation densities by [CY20] and [LZ22a, Theorem 3.5.1] (see Proposition 6.15 below).

Now, we consider $\mathrm{Den}'(I_{n,h},T)-(-q)^{nh}\mathrm{Den}'(I_n,T)$. Then, it is possible to write the difference between these two in terms of a certain linear sum of representation densities (see (6.7)). Therefore, we can use the linear sum expression of $(-q)^{nh}\mathrm{Den}'(I_n,T)$ and the difference $\mathrm{Den}'(I_{n,h},T)-(-q)^{nh}\mathrm{Den}'(I_n,T)$ to find all constants $D_{n,h}(\lambda)$. This is what we will do in the next few pages.

Definition 6.11. (1) For $Y\in\mathcal{R}_n$, we define $t_0(Y)=|\{e_i\mid e_i\ge 0\}|$, and for $k\ge 1$, we define $t_k(Y)=|\{e_i\mid e_i=-k\}|$.
(2) For $\eta\in\mathcal{R}_n^{0+}$, and $k\ge 0$, we define $t_k(\eta)=|\{\eta_i\mid \eta_i=k\}|$, and $t_{\ge k}(\eta)=|\{\eta_i\mid \eta_i\ge k\}|$.
(3) For $Y\in\mathcal{R}_n$, we define $t(Y)=(t_0(Y),t_1(Y),\dots)$. Similarly, for $\eta\in\mathcal{R}_n^{0+}$, we define $t(\eta)=(t_0(\eta),t_1(\eta),\dots)$.

Definition 6.12. (1) For $\lambda=(\overline{\lambda},\overbrace{1,\dots,1}^{t_1(\lambda)},\overbrace{0,\dots,0}^{t_0(\lambda)})\in\mathcal{R}_n^{0+}$, and $0\le s\le t_0(\lambda)$, we define $\lambda_s^+\coloneqq(\overline{\lambda},\overbrace{1,\dots,1}^{t_1(\lambda)+s},\overbrace{0,\dots,0}^{t_0(\lambda)-s})$, by replacing s zeros by s 1's.

(2) For $\lambda = (\overline{\lambda}, \overbrace{1, \dots, 1}^{t_1(\lambda)}, \overbrace{0, \dots, 0}^{t_0(\lambda)}) \in \mathcal{R}_n^{0+}$ and $0 \le s \le t_1(\lambda)$, we define $\lambda_s^- = (\overline{\lambda}, \overbrace{1, \dots, 1}^{t_1(\lambda)-s}, \overbrace{0, \dots, 0}^{t_0(\lambda)+s})$, by replacing s 1's by s zeros.

(3) For $0 \le l \le n$ and $\lambda \in \mathcal{R}_n^{0+}$ such that $t_0(\lambda) \ge l$, we define $\lambda^{\vee_l}$ as the element in $\mathcal{R}_{n-l}^{0+}$ such that $\lambda = (\lambda^{\vee_l}, 0, \dots, 0)$.

Definition 6.13. Assume that $k \ge 0$, $\alpha, \eta \in \mathcal{R}_n^{0+}$, $Y \in \mathcal{R}_n^{0+}$, and $t(\eta) = t(Y)$.

(1) We define $\mathcal{B}_k(Y) = \sum_i \min(0, e_i + k) - \min(0, e_i)$, and $\mathcal{B}_k(\eta) = \sum_i \min(k, \eta_i)$.

(2) We define $\mathcal{B}_\alpha(Y) = \sum_i \mathcal{B}_{\alpha_i}(Y)$, and $\mathcal{B}_\alpha(\eta) = \sum_i \mathcal{B}_{\alpha_i}(\eta)$.

(3) We define $f(Y)$ by $f(Y) = \prod_i (-q)^{n \min(0, e_i)}$.

By Lemma 5.3 and the fact that $\mathcal{R}_n$ forms a complete set of representatives of $\Gamma_n \backslash X_n(F)$, we have that for $\lambda \in \mathcal{R}_n^{0+}$,

$$\mathrm{Den}'(I_{n,h}, B) = \sum_{Y \in \mathcal{R}_n} \frac{\mathcal{G}(Y, B)\mathcal{F}'(Y, I_{n,h})}{\alpha(Y; \Gamma_n)}, \tag{6.1}$$

$$\mathrm{Den}'(I_n, B) = \sum_{Y \in \mathcal{R}_n} \frac{\mathcal{G}(Y, B)\mathcal{F}'(Y, I_n)}{\alpha(Y; \Gamma_n)}, \tag{6.2}$$

$$\mathrm{Den}(A_\lambda, B) = \sum_{Y \in \mathcal{R}_n} \frac{\mathcal{G}(Y, B)\mathcal{F}(Y, A_\lambda)}{\alpha(Y; \Gamma_n)}. \tag{6.3}$$

By [Cho23, Lemma 3.2, Section 3.2], we know that $\mathcal{F}'(Y, I_{n,h})$ can be written uniquely as a linear sum of $\mathcal{F}(Y, A_\lambda)$, $\lambda \in \mathcal{R}_n^{0+}$. As in the proof of [Cho22b, Lemma 3.15], for $Y = Y_{\sigma,e} = \sigma \mathrm{Diag}(\pi^{e_1}, \dots, \pi^{e_n})$, we have that (cf. [Cho23, (4.1.1)])

$$\mathcal{F}'(Y, I_{n,h}) - (-q)^{hn}\mathcal{F}'(Y, I_n) = \begin{cases} 0 & \text{if } t_0(Y) = 0, \\ (\sum_j \min(0, e_j)) f(Y)((-q)^{h(n-1)} - (-q)^{hn}) & \text{if } t_0(Y) = 1, \\ \vdots & \\ (\sum_j \min(0, e_j)) f(Y)((-q)^{h(n-k)} - (-q)^{hn}) & \text{if } t_0(Y) = k, \\ \vdots & \\ (\sum_j \min(0, e_j)) f(Y)(1 - (-q)^{hn}) & \text{if } t_0(Y) = n. \end{cases}$$

Now, let us define the following constants and matrices.

Definition 6.14.

(1) (cf. [Cho23, Lemma 4.4]) For $0 \le i \le l$, we define constants d_{il} by

$$d_{il} = (-q)^{-in} \prod_{\substack{0 \le m \le l \\ m \ne i}} \frac{1}{((-q)^{-i} - (-q)^{-m})}.$$

(2) We define the upper triangular $(n+1)\times(n+1)$ matrix Δ by $\Delta = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ 0 & d_{11} & \dots & d_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}$.

(3) For $l \leq 0$, we define $\mathfrak{M}_l$ by $\mathfrak{M}_l : \begin{pmatrix} 1 & (-q)^{2n} & \dots & (-q)^{2ln} \\ 1 & (-q)^{2n-1} & \dots & (-q)^{l(2n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (-q)^{2n-l} & \dots & (-q)^{l(2n-l)} \end{pmatrix}$.

(4) For $0 \leq i, j \leq n$, we define constants $\mathcal{A}_{ij}$ by $(\mathcal{A}_{ij})_{0\leq i,j\leq n} = \mathfrak{M}_n\Delta$.

(5) For $0 \leq i \leq n$, we define constants $\mathcal{K}_i$ by $(\mathcal{A}_{ij}) \begin{pmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \vdots \\ \mathcal{K}_n \end{pmatrix} = \begin{pmatrix} 0 \\ (-q)^{h(n-1)} - (-q)^{hn} \\ \vdots \\ 1 - (-q)^{hn} \end{pmatrix}$.

Proposition 6.15. [Cho23, Proposition 3.3, Proposition 3.4] *We define the following constants.*

(1) For $\alpha \in \mathcal{R}_n^{0+}$ such that $\sum \alpha_i =$odd, we write $\mathcal{C}_\alpha$ for $\mathcal{C}_\alpha = \prod_{i=1}^{t_{\geq 1}(\alpha)-1}(1-(-q)^i)$. Here, we define $\mathcal{C}_\alpha = 1$ if $t_{\geq 1}(\alpha) = 1$.

(2) For $\alpha \in \mathcal{R}_n^{0+}$ such that $\sum \alpha_i =$even and $\alpha \neq (0,0,\dots,0)$, we write $\mathcal{C}_\alpha$ for $\mathcal{C}_\alpha = -\prod_{i=1}^{t_{\geq 1}(\alpha)-1}(1-(-q)^i)$. Here, we define $\mathcal{C}_\alpha = -1$ if $t_{\geq 1}(\alpha) = 1$. Also, if $\alpha = (0,0,\dots,0)$, we define $\mathcal{C}_\alpha = \dfrac{\mathrm{Den}'(I_n, I_n)}{\mathrm{Den}(I_n, I_n)}$.

Then, we have that $\dfrac{\mathrm{Den}'(I_n, B)}{\mathrm{Den}(I_n, I_n)} = \sum_{\alpha\in\mathcal{R}_n^{0+}} \mathcal{C}_\alpha \dfrac{\mathrm{Den}(A_\alpha, B)}{\mathrm{Den}(A_\alpha, A_\alpha)}$.

Using Definition 6.14 and Proposition 6.15, and following [Cho23, Section 4], we have the following equation.

(6.4)
$$\mathcal{F}'(Y, I_{n,h}) - (-q)^{hn}\mathcal{F}'(Y, I_n) = \sum_{l=0}^{n} \mathcal{K}_l \{ \sum_{\lambda\in\mathcal{R}_n^{0+}, t_0(Y)\geq l} \mathcal{C}_{\lambda^{\vee_l}} \frac{\mathrm{Den}(I_{n-l}, I_{n-l})}{\mathrm{Den}(A_{\lambda^{\vee_l}}, A_{\lambda^{\vee_l}})} \sum_{0\leq i\leq l} d_{il}\mathcal{F}_0(Y, A_{\lambda_i^+}) \}.$$

Now, we have the following lemma.

Lemma 6.16. *(cf. [Cho23, Lemma 4.5]) For $i = 0$ and $h+1 \leq i \leq n$, $\mathcal{K}_i = 0$. Also, $d_{hh}\mathcal{K}_h = 1$ and for $1 \leq l \leq h-1$, we have*

$$d_{h-l,h-l}\mathcal{K}_{h-l} = (-q)^n \frac{1-(-q)^{-h+l-1}}{1-(-q)^{-l}} d_{h-l+1,h-l+1}\mathcal{K}_{h-l+1}.$$

Proof. The proof is almost identical to the proof of [Cho23, Lemma 4.5]. □

Now, we are ready to write the following proposition on the constants $D_{n,h}(\lambda)$.

Proposition 6.17. *(cf. [Cho23, Theorem 4.7]) For $0 \leq h \leq n$ and $\lambda \in \mathcal{R}_n^{0+}$ such that $\lambda \neq (1^t, 0^n - t), t \leq h-1$, we have*

(6.5)

$$\begin{aligned} D_{n,h}(\lambda) =& \frac{q^{h(n-h)}\prod_{l=1}^{n}(1-(-q)^{-l})}{\prod_{l=1}^{h}(1-(-q)^{-l})\prod_{l=1}^{n-h}(1-(-q)^{-l})}\mathcal{C}_{\lambda} \\ &+ \sum_{\substack{1\le i\le h \\ \max\{i-t_0(\lambda),0\}\le s \\ \le \min\{i,t_1(\lambda)\}}} (-q)^{n(h-i)+(i-s)(2n-i+s+1)/2-h^2+s(2n-2t_0(\lambda)-s)}(-1)^{i+h} \\ &\quad \times \frac{\prod_{l=1}^{n-i}(1-(-q)^{-l})}{\prod_{l=1}^{n-h}(1-(-q)^{-l})\prod_{l=1}^{h}(1-(-q)^{-l})} \times \frac{\prod_{l=s+1}^{h}(1-(-q)^{-l})}{\prod_{l=1}^{h-i}(1-(-q)^{-l})\prod_{l=1}^{i-s}(1-(-q)^{-l})} \\ &\quad \times \frac{\prod_{l=1}^{l=t_0(\lambda)}(1-(-q)^{-l})\prod_{l=1}^{l=t_1(\lambda)}(1-(-q)^{-l})}{\prod_{l=1}^{l=t_0(\lambda)-i+s}(1-(-q)^{-l})\prod_{l=1}^{l=t_1(\lambda)-s}(1-(-q)^{-l})} \times \mathcal{C}_{(\lambda_s^-)^{\vee_i}}. \end{aligned}$$

Here, we choose the following convention: for $k \le 0$, we assume that $\prod_{l=1}^{k}(*) = 1$ and $\sum_{l=1}^{k}(*) = 0$. If $\lambda = (1^t, 0^{n-t}), t \le h-1$, then $D_{n,h}(\lambda) =$ the right hand side of (6.5) $+ \dfrac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1-(-q)^{-(h-t)}}$. In particular, $D_{n,h}(\lambda)$ depends only on n, $t_0(\lambda)$, $t_1(\lambda)$, and the parity of $\sum_i \lambda_i$.

Remark 6.18. Note that when $h = 0$, the sum $\sum_{\substack{1\le i\le h \\ \max\{i-t_0(\lambda),0\}\le s\le \min\{i,t_1(\lambda)\}}} (*)$ in (6.5) is empty, and hence $D_{n,0}(\lambda) = \mathcal{C}_{\lambda}$ which is obvious by construction. Therefore, Proposition 6.17 decomposes $D_{n,h}$ into a weighted summation of $D_{n,0}$, which is the Cho-Yamauchi constant in the good reduction case.

Proof of Proposition 6.17. The proof is almost identical to the proof of [Cho23, Theorem 4.7]; therefore, let me just write which parts are different. First, note that for $B \in X_n(F)$ of the form in (5.1), Proposition 6.1 implies that

$$\frac{W'_{n-h,n}(B)}{W_{n,n}(A_n,1)} = \frac{q^{-4n^2+(n+h)(n-h)}\mathrm{Den}(\pi A_n, I_{n-h})\mathrm{Den}(I_{n+h,h}, I_h)\mathrm{Den}'(I_{n,h},T)}{W_{n,n}(A_n,1)}. \tag{6.6}$$

Now, by (6.1), (6.2), (6.3), and (6.4), we have

(6.7)

$$\mathrm{Den}'(I_{n,h},T) = (-q)^{nh}\mathrm{Den}'(I_n,T) + \sum_{l=0}^{n}\mathcal{K}_l\{\sum_{\lambda\in\mathcal{R}_n^{0+},t_0(Y)\ge l}\mathcal{C}_{\lambda^{\vee_l}}\frac{\mathrm{Den}(I_{n-l},I_{n-l})}{\mathrm{Den}(A_{\lambda^{\vee_l}},A_{\lambda^{\vee_l}})}\sum_{0\le i\le l}d_{il}\mathrm{Den}(A_{\lambda_i^+},T)\}.$$

Now, we can easily follow the proof of [Cho23, Theorem 4.7] by using Lemma 6.16, (6.6), and (6.7). Also, for $\lambda = (1^t, 0^{n-t}), t \le h-1$, we use Proposition 6.6. □

Note that $D_{n,h}(\lambda)$ depends only on $t_{\ge 2}(\lambda)$, $t_1(\lambda)$, $t_0(\lambda)$, and the parity of $\sum_i \lambda_i$. Also, by Remark 6.4, we only need to consider λ such that $\sum_i \lambda_i \equiv h+1 \pmod 2$. Also, later on, we will establish inductive formulas relating $D_{n,h}(\lambda)$ for λ with different $t_{\ge 2}(\lambda), t_1(\lambda)$ and $t_0(\lambda)$. So we introduce the following notation to streamline the computation.

Definition 6.19. Assume that $h, n, a, b, c, i, j, s \ge 0$, $0 \le s \le i \le h \le n$, $0 \le j \le n$, and $n = a+b+c$.

(1) If $c \neq n$, we write $\mathcal{C}_j(a,b,c)$ for $\mathcal{C}_j(a,b,c) = (-1)^{j+1}\prod_{i=1}^{a+b-1}(1-(-q)^i)$. Also, if $c = n$, we define $\mathcal{C}_j(0,0,n) = \dfrac{\mathrm{Den}'(I_n, I_n)}{\mathrm{Den}(I_n, I_n)}$. Then, for $\lambda \in \mathcal{R}_n^{0+}$, we have that $\mathcal{C}_\lambda = \mathcal{C}_{|\lambda|}(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$.

(2) Let $M_{n,h}(a,b,c,i,s)$ be the constant

$$\begin{aligned} M_{n,h}(a,b,c,i,s) =& (-q)^{n(h-i)+(i-s)(2n-i+s+1)/2-h^2+s(2n-2c-s)}(-1)^{i+h} \\ &\times \frac{\prod_{l=1}^{n-i}(1-(-q)^{-l})}{\prod_{l=1}^{n-h}(1-(-q)^{-l})\prod_{l=1}^{h}(1-(-q)^{-l})} \times \frac{\prod_{l=s+1}^{h}((1-(-q)^{-l})}{\prod_{l=1}^{h-i}(1-(-q)^{-l})\prod_{l=1}^{i-s}(1-(-q)^{-l})} \\ &\times \frac{\prod_{l=1}^{c}(1-(-q)^{-l})\prod_{l=1}^{b}(1-(-q)^{-l})}{\prod_{l=1}^{c-i+s}(1-(-q)^{-l})\prod_{l=1}^{b-s}(1-(-q)^{-l})} \times \mathcal{C}_{h+1-s}(a, b-s, c+s-i). \end{aligned}$$

(3) For $(a,b,c) \neq (0,t,n-t), t \leq h-1$, we write $D_{n,h}(a,b,c)$ for

$$D_{n,h}(a,b,c) = \sum_{\substack{0\leq i\leq h \\ \max(i-c,0)\leq s\leq \min(i,b)}} M_{n,h}(a,b,c,i,s) = \sum_{\substack{0\leq s\leq \min(h,b) \\ s\leq i\leq \min(s+c,h)}} M_{n,h}(a,b,c,i,s).$$

(4) For $(a,b,c) = (0,t,n-t), t \leq h-1$, $t \equiv h+1 \pmod 2$, we write $D_{n,h}(a,b,c)$ for

$$D_{n,h}(0,t,n-t) = \sum_{\substack{0\leq s\leq \min(h,b) \\ s\leq i\leq \min(s+c,h)}} M_{n,h}(a,b,c,i,s) + \frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1-(-q)^{-(h-t)}}.$$

Proposition 6.20. *For $\lambda \in \mathcal{R}_n^{0+}$, we have that*

$$D_{n,h}(\lambda) = D_{n,h}(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)).$$

Proof. This follows from the definitions of $M_{n,h}(a,b,c,i,s)$, $D_{n,h}(a,b,c)$, Proposition 6.17, and the fact that

$$M_{n,h}(a,b,c,0,0) = \frac{q^{h(n-h)}\prod_{l=1}^{n}(1-(-q)^{-l})}{\prod_{l=1}^{h}(1-(-q)^{-l})\prod_{l=1}^{n-h}(1-(-q)^{-l})}\mathcal{C}_{h+1}(a,b,c).$$

□

7. Inductive relations among Cho-Yamauchi constants

In this section, we will prove some inductive relations among $D_{n,h}(\lambda)$. The main result of this section can be summarized as follows: Let $0 \leq a,b,c,h, \leq n$, $a+b+c = n$. Assume further that $(a-1, b+1, c) \neq (0, h, n-h)$. Then, we have the following result.

(1) (Theorem 7.4) If $c \leq n-h$ and $a \geq 1$, then

$$D_{n,h}(a,b,c) - D_{n,h}(a-1,b+1,c) = -(-q)^{2n-h-1-b-2c}D_{n-1,h-1}(a-1,b,c).$$

(2) (Lemma 7.5, Proposition 7.6) If $c > n-h$, then $D_{n,h}(a,b,c) = 0$.

(3) (Theorem 7.8 (1)) Assume that $a = 0$ and $h+1 \leq b$. Then, we have

$$D_{n,h}(0,b,c) = \frac{\prod_{l=h+1}^{b}(1-(-q)^l)}{(1-(-q)^{b-h})}.$$

(4) (Theorem 7.8 (2)) Assume that $a = 1$ and $h-1 \leq b$. Then, we have

$$D_{n,h}(1,b,c) = \begin{cases} 1 & \text{if } b = h-1, h; \\ \prod_{l=h+1}^{b}(1-(-q)^l) & \text{if } b \geq h+1. \end{cases}$$

Here, the main reason for excluding the case $(a-1,b+1,c)=(0,h,n-h)$ is because it is not necessary by Remark 6.4. First, let us prove the following lemma.

Lemma 7.1. *Let $0\leq a,b,c,h\leq n$, $a+b+c=n$, and $0\leq s\leq \min(h,b)$.*

(1) If $c\leq n-h$ and $(a,b,c)\neq(0,h,n-h)$, we have that

$$\begin{aligned}&\sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s)\\ &=(-1)^{n+c+s+1}(-q)^{\frac{c}{2}+\frac{c^2}{2}-h^2-\frac{n}{2}-cn+hn+\frac{n^2}{2}-ch+\frac{s}{2}-\frac{s^2}{2}}\\ &\quad\times\frac{\prod_{l=1}^{b}(1-(-q)^{-l})\prod_{l=1}^{n-c-s}(1-(-q)^{-l})\prod_{l=1}^{n-c-s-1}(1-(-q)^{-l})}{\prod_{l=1}^{b-s}(1-(-q)^{-l})\prod_{l=1}^{s}(1-(-q)^{-l})\prod_{l=1}^{n-h-c}(1-(-q)^{-l})\prod_{l=1}^{h-s}(1-(-q)^{-l})}.\end{aligned}$$

(2) If $c>n-h$ and $a\neq 0$, we have that $\sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s)=0$.

Proof. By our assumption, we have that $(a,b-s,c+s-i)\neq(0,0,n-i)$ for any $i\leq s\leq\min(s+c,h)$. Therefore, by Definition 6.19 (1), we have that

$$\begin{aligned}\mathcal{C}_{h+1-s}(a,b-s,c+s-i)&=(-1)^{h-s}\textstyle\prod_{l=1}^{n-c-s-1}(1-(-q)^{l})\\&=(-1)^{n+c+h+1}(-q)^{(n-c-s-1)(n-c-s)/2}\textstyle\prod_{l=1}^{n-c-s-1}(1-(-q)^{-l}),\end{aligned}$$

which is independent of i. Also, by Definition 6.19 (2), we have that

$$M_{n,h}(a,b,c,i,s)=\mathcal{C}_{h+1-s}(a,b-s,c+s-i)Y_{n,h}(b,c,s)X_{n,h}(c,i,s),$$

where

$$Y_{n,h}(b,c,s):=(-1)^h(-q)^{-h^2+hn-\frac{s}{2}-\frac{3s^2}{2}-2cs+ns}\frac{\prod_{l=1}^{c}(1-(-q)^{-l})\prod_{l=1}^{b}(1-(-q)^{-l})}{\prod_{l=1}^{n-h}(1-(-q)^{-l})\prod_{l=1}^{s}(1-(-q)^{-l})\prod_{l=1}^{b-s}(1-(-q)^{-l})},$$

$$X_{n,h}(c,i,s):=\frac{(-1)^i(-q)^{\frac{i}{2}-\frac{i^2}{2}+is}\prod_{l=1}^{n-i}(1-(-q)^{-l})}{\prod_{l=1}^{h-i}(1-(-q)^{-l})\prod_{l=1}^{i-s}(1-(-q)^{-l})\prod_{l=1}^{c-i+s}(1-(-q)^{-l})}.$$

Now, we note that Lemma 7.1 follows from the following claim

(7.1)

$$\sum_{i=s}^{\min(s+c,h)} X_{n,h}(c,i,s)=\begin{cases}\dfrac{(-1)^s(-q)^{-ch+cs+\frac{s}{2}+\frac{s^2}{2}}\prod_{l=1}^{n-h}(1-(-q)^{-l})\prod_{l=1}^{n-s-c}(1-(-q)^{-l})}{\prod_{l=1}^{h-s}(1-(-q)^{-l})\prod_{l=1}^{c}(1-(-q)^{-l})\prod_{l=1}^{n-h-c}(1-(-q)^{-l})} & \text{if } c\leq n-h,\\ 0 & \text{if } c>n-h.\end{cases}$$

Therefore, it suffices to show that (7.1) holds.

If $h=s$ or $c=0$, then (7.1) holds by direct computation.

Now, assume that $c \geq 1$ and $h > s$. In Lemma 7.2 below, we will prove that

$$\sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} = \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \sum_{i=s}^{\min(s+c-1,h)} \frac{X_{n-1,h}(c-1,i,s)}{X_{n-1,h}(c-1,s,s)}. \tag{7.2}$$

Assume (7.2) holds for now. Then we are ready to prove (7.1).

If $c \leq n-h$, by applying (7.2) repeatedly, we have that

$$\begin{aligned}\sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} &= (-q)^{-c(h-s)} \frac{\prod_{l=n-h-c+1}^{n-h}(1-(-q)^{-l})}{\prod_{l=n-s-c+1}^{n-s}(1-(-q)^{-l})} \sum_{i=s}^{\min(s+0,h)} \frac{X_{n-c,h}(0,i,s)}{X_{n-c,h}(0,s,s)} \\ &= (-q)^{-c(h-s)} \frac{\prod_{l=n-h-c+1}^{n-h}(1-(-q)^{-l})}{\prod_{l=n-s-c+1}^{n-s}(1-(-q)^{-l})}.\end{aligned}$$

Combining this with $X_{n,h}(c,s,s) = \dfrac{(-1)^s(-q)^{\frac{s}{2}+\frac{s^2}{2}} \prod_{l=1}^{n-s}(1-(-q)^{-l})}{\prod_{l=1}^{h-s}(1-(-q)^{-l}) \prod_{l=1}^{c}(1-(-q)^{-l})}$, we see that (7.1) holds when $c \leq n-h$.

Now, assume that $c > n-h$. By (7.2), we have that

$$\sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} = (-q)^{-(n-h)(h-s)} \frac{\prod_{l=1}^{n-h}(1-(-q)^{-l})}{\prod_{l=h-s+1}^{n-s}(1-(-q)^{-l})} \sum_{i=s}^{\min(s+c-n+h,h)} \frac{X_{h,h}(c-n+h,i,s)}{X_{h,h}(c-n+h,s,s)}.$$

Therefore, it suffices to show that $\displaystyle\sum_{i=s}^{\min(s+c-n+h,h)} \frac{X_{h,h}(c-n+h,i,s)}{X_{h,h}(c-n+h,s,s)} = 0.$

Note that $s \leq \min(h,b)$ and hence $s+c \leq b+c \leq n$. Therefore, $\min(s+c-n+h,h) = s+c-n+h$. For simplicity, let us write $c' = c-n+h \geq 1$.

We claim that

$$\sum_{i=s}^{s+k} \frac{X_{h,h}(c',i,s)}{X_{h,h}(c',s,s)} = (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c'-k}^{c'-1}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})}, \tag{7.3}$$

which specializes to $\displaystyle\sum_{i=s}^{s+c'} \frac{X_{h,h}(c',i,s)}{X_{h,h}(c',s,s)} = 0$, when $k = c'$.

Therefore it suffices to prove (7.3). We prove this by induction on k. By definition, we have

$$\frac{X_{h,h}(c',s+k,s)}{X_{h,h}(c',s,s)} = (-1)^k (-q)^{-\frac{k(k-1)}{2}} \frac{\prod_{l=c'-k+1}^{c'}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})}.$$

When $k=0$, both sides are 1. When $k=1$, $\displaystyle\sum_{i=s}^{s+1} \frac{X_{h,h}(c',i,s)}{X_{h,h}(c',s,s)} = -(-q)^{-1} \frac{1-(-q)^{-(c'-1)}}{1-(-q)^{-1}}$. Now, assume that (7.3) holds for k. Then, we have

$$\begin{aligned}\sum_{i=s}^{s+k+1} \frac{X_{h,h}(c',i,s)}{X_{h,h}(c',s,s)} &= (-1)^k (-q)^{-\frac{k(k+1)}{2}} \frac{\prod_{l=c'-k}^{c'-1}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})} + \frac{X_{h,h}(c',s+k+1,s)}{X_{h,h}(c',s,s)} \\ &= (-1)^{k+1} (-q)^{-\frac{(k+1)(k+2)}{2}} \frac{\prod_{l=c'-k-1}^{c'-1}(1-(-q)^{-l})}{\prod_{l=1}^{k+1}(1-(-q)^{-l})}.\end{aligned}$$

This shows that (7.3) holds and finishes the proof of (7.1), and hence the proof of the lemma. □

Lemma 7.2. *For $c \geq 1$ and $h > s$, the function $X_{n,h}(c,i,s)$ (defined in the proof of the Lemma 7.1) satisfies*

$$\sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} = \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \sum_{i=s}^{\min(s+c-1,h)} \frac{X_{n-1,h}(c-1,i,s)}{X_{n-1,h}(c-1,s,s)}. \tag{7.4}$$

Proof. To prove (7.4), we define $Z_{n,h}(c,j,s)$ to be

$$Z_{n,h}(c,j,s) = (-1)^{j-s}(-q)^{-\frac{(j-s)(j-s+1)}{2}} \frac{\prod_{l=c-j+s}^{c-1}(1-(-q)^{-l}) \prod_{l=h-j+1}^{h-s}(1-(-q)^{-l})}{\prod_{l=1}^{j-s}(1-(-q)^{-l}) \prod_{l=n-j+1}^{n-s}(1-(-q)^{-l})}.$$

Note that if $c \geq 1$ and $h > s$, we have that $\min(s+c,h) > s$.

Now, we claim the following statement: for $k \geq 1$, we have

$$\sum_{i=s}^{s+k} \frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} - \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \sum_{i=s}^{s+k-1} \frac{X_{n-1,h}(c-1,i,s)}{X_{n-1,h}(c-1,s,s)} = Z_{n,h}(c,s+k,s). \tag{7.5}$$

By definition, we have that

$$\frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} = (-1)^{i-s}(-q)^{-\frac{(i-s)(i-s-1)}{2}} \frac{\prod_{l=c-i+s+1}^{c}(1-(-q)^{-l}) \prod_{l=h-i+1}^{h-s}(1-(-q)^{-l})}{\prod_{l=1}^{i-s}(1-(-q)^{-l}) \prod_{l=n-i+1}^{n-s}(1-(-q)^{-l})}.$$

If $k=1$, (7.5) holds by direct computation.

Now, assume that (7.5) is true for k, then it suffices to show that

$$\begin{aligned} &Z_{n,h}(c,s+k,s) + \frac{X_{n,h}(c,s+k+1,s)}{X_{n,h}(c,s,s)} \\ &= \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \frac{X_{n-1,h}(c-1,s+k,s)}{X_{n-1,h}(c-1,s,s)} + Z_{n,h}(c,s+k+1,s). \end{aligned} \tag{7.6}$$

Indeed, we can show that (7.6) holds by direct computation, and hence (7.5) holds.

Now, we are ready to prove (7.4). When $h \geq s+c$, we have that $\min(s+c,h) = s+c$ and $\min(s+c-1,h) = s+c-1$. Also, note that $Z_{n,h}(c,s+c,s) = 0$. Now, (7.5) implies that (7.4) holds when $h \geq s+c$.

When $h \leq s+c-1$, we have that $\min(s+c,h) = h$ and $\min(s+c-1,h) = h$. Therefore, by (7.5), we have

$$\begin{aligned} &\textstyle\sum_{i=s}^{\min(s+c,h)} \frac{X_{n,h}(c,i,s)}{X_{n,h}(c,s,s)} - \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \sum_{i=s}^{\min(s+c-1,h)} \frac{X_{n-1,h}(c-1,i,s)}{X_{n-1,h}(c-1,s,s)} \\ &= Z_{n,h}(c,h,s) - \frac{(-q)^{-(h-s)}(1-(-q)^{-(n-h)})}{1-(-q)^{-(n-s)}} \frac{X_{n-1,h}(c-1,h,s)}{X_{n-1,h}(c-1,s,s)} = 0. \end{aligned}$$

This finishes the proof of the lemma. □

Lemma 7.3. *Let $0 \leq a,b,c,h \leq n$, $a+b+c = n$, and $0 \leq s \leq \min(h,b)$. If $c \leq n-h$ and $(a,b,c) \neq (0,h,n-h)$, we have that*

$$\begin{aligned} &\textstyle\sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s) - \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a-1,b+1,c,i,s) \\ &= -(-q)^{2n-h-1-b-2c} \textstyle\sum_{i=s-1}^{\min(s+c-1,h-1)} M_{n-1,h-1}(a-1,b,c,i,s-1). \end{aligned}$$

Proof. This follows from Lemma 7.1 (1). □

Theorem 7.4. *Let $0 \le a, b, c, h \le n$ and $a+b+c = n$. If $c \le n-h$, $1 \le a$, and $(a-1, b+1, c) \ne (0, h, n-h)$, we have that*

$$D_{n,h}(a,b,c) - D_{n,h}(a-1,b+1,c) = -(-q)^{2n-h-1-b-2c} D_{n-1,h-1}(a-1,b,c).$$

Proof. First, by Lemma 7.1, we have that

$$\sum_{i=0}^{\min(0+c,h)} M_{n,h}(a,b,c,i,0) = (-1)^{n+c+1}(-q)^{\frac{c}{2}+\frac{c^2}{2}-h^2-\frac{n}{2}-cn+hn+\frac{n^2}{2}-ch} \frac{\prod_{l=1}^{n-c}(1-(-q)^{-l})\prod_{l=1}^{n-c-1}(1-(-q)^{-l})}{\prod_{l=1}^{n-h-c}(1-(-q)^{-l})\prod_{l=1}^{h}(1-(-q)^{-l})}.$$

Since this does not depend on a, b, we have that $\sum_{i=0}^{\min(0+c,h)} M_{n,h}(a,b,c,i,0) = \sum_{i=0}^{\min(0+c,h)} M_{n,h}(a-1,b+1,c,i,0)$. Since $D_{n,h}(a,b,c) = \sum_{s=0}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s)$, we have that

$$\begin{aligned} &D_{n,h}(a,b,c) - D_{n,h}(a-1,b+1,c) \\ &= \sum_{s=1}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s) - \sum_{s=1}^{\min(h,b+1)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a-1,b+1,c,i,s). \end{aligned}$$

Now, assume that $h \le b$. Then $\min(h,b) = h$, $\min(h,b+1) = h$, $\min(h-1,b) = h-1$. Therefore, by Lemma 7.3, we have

$$\begin{aligned} &D_{n,h}(a,b,c) - D_{n,h}(a-1,b+1,c) = \sum_{s=1}^{h}\{ \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s) - \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a-1,b+1,c,i,s)\} \\ &= \sum_{s=0}^{h-1} -(-q)^{2n-h-1-b-2c} \sum_{i=s}^{\min(s+c,h-1)} M_{n-1,h-1}(a-1,b,c,i,s) = -(-q)^{2n-h-1-b-2c} D_{n-1,h-1}(a-1,b,c). \end{aligned}$$

This finishes the proof of the theorem when $h \le b$.

Now, assume that $h > b$. Then $\min(h,b) = b$, $\min(h,b+1) = b+1$. Therefore, by Lemma 7.3, we have that

(7.7)

$$\begin{aligned} &D_{n,h}(a,b,c) - D_{n,h}(a-1,b+1,c) \\ &= \sum_{s=0}^{b-1} -(-q)^{2n-h-1-b-2c} \sum_{i=s}^{\min(s+c,h-1)} M_{n-1,h-1}(a-1,b,c,i,s) - \sum_{i=b+1}^{\min(b+1+c,h)} M_{n,h}(a-1,b+1,c,i,b+1). \end{aligned}$$

By Lemma 7.1, we have that

$$\begin{aligned} &- \sum_{i=b+1}^{\min(b+1+c,h)} M_{n,h}(a-1,b+1,c,i,b+1) \\ &= -(-1)^{n-1+c+b+1}(-q)^{2n-h-1-b-2c+\frac{c}{2}+\frac{c^2}{2}-(h-1)^2-\frac{n-1}{2}-c(n-1)+(h-1)(n-1)+\frac{(n-1)^2}{2}-c(h-1)+\frac{b}{2}-\frac{b^2}{2}} \\ &\quad \times \frac{\prod_{l=1}^{n-1-c-b}(1-(-q)^{-l})\prod_{l=1}^{n-1-c-b-1}(1-(-q)^{-l})}{\prod_{l=1}^{(n-1)-(h-1)-c}(1-(-q)^{-l})\prod_{l=1}^{h-1-b}(1-(-q)^{-l})} \\ &= -(-q)^{2n-h-1-b-2c} \sum_{i=b}^{\min(b+c,h-1)} M_{n-1,h-1}(a-1,b,c,i,b). \end{aligned}$$

Therefore, since $\min(h-1,b)=b$, (7.7) can be written as

$$D_{n,h}(a,b,c)-D_{n,h}(a-1,b+1,c)=-(-q)^{2n-h-1-b-2c}D_{n-1,h-1}(a-1,b,c).$$

This finishes the proof of the theorem. □

Lemma 7.5. *Let $0\leq a,b,c,h\leq n$, and $a+b+c=n$. If $c>n-h$ and $a\neq 0$, we have that*

$$D_{n,h}(a,b,c)=0.$$

Therefore, for $\lambda\in\mathcal{R}_n^{0+}$ such that $t_0(\lambda)>n-h$, and $t_{\geq 2}(\lambda)\neq 0$, we have that

$$D_{n,h}(\lambda)=0.$$

Proof. This follows from Lemma 7.1 (2), Definition 6.19 (3) and Proposition 6.20. □

Proposition 7.6. *For $0\leq t\leq h-1$, $t\equiv h+1 \pmod 2$, and $\lambda=(1^t,0^{n-t})\in\mathcal{R}_n^{0+}$, we have that*

$$D_{n,h}(\lambda)=0.$$

Proof. First, by Proposition 6.20, we have that

$$D_{n,h}(\lambda)=D_{n,h}(0,t,n-t)=\sum_{\substack{0\leq s\leq\min(h,b)\\ s\leq i\leq\min(s+c,h)}}M_{n,h}(a,b,c,i,s)+\frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1-(-q)^{-(h-t)}}.$$

Before we start the proof of the proposition, let us say why we consider this case separately. Recall from Definition 6.19 (1) that $\mathcal{C}_l(0,0,k)$ is defined separately. If λ is not of the form $(1^t,0^{n-t})$, $t<h$, these constants $\mathcal{C}_l(0,0,k)$'s do not appear in the sum $\sum_{\substack{0\leq s\leq\min(h,b)\\ s\leq i\leq\min(s+c,h)}}M_{n,h}(a,b,c,i,s)$. However, if $\lambda=(1^t,0^{n-t})$, $t<h$, $M_{n,h}(0,t,n-t,i,t)$ has the term $\mathcal{C}_{h+1-t}(0,0,n-i)$. Therefore, we need to be careful when we compute $D_{n,h}(0,t,n-t)$.

Now, let us start the proof of the proposition. Recall from the proof of the Lemma 7.1 that we have $M_{n,h}(a,b,c,i,s)=\mathcal{C}_{h+1-s}(a,b-s,c+s-i)Y_{n,h}(b,c,s)X_{n,h}(c,i,s)$.

Also, by (7.1), we have that

$$\textstyle\sum_{i=s}^{\min(s+n-t,h)}Y_{n,h}(t,n-t,s)X_{n,h}(n-t,i,s)=Y_{n,h}(t,n-t,s)\sum_{i=s}^{\min(s+n-t,h)}X_{n,h}(n-t,i,s)=0$$

Also, note that $\min(h,t)=t$ and $\mathcal{C}_{h+1-s}(0,t-s,n-t+s-i)$ does not depend on i if $s\neq t$. Therefore, we have that

$$D_{n,h}(0,t,n-t)-\frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1-(-q)^{-(h-t)}}=\sum_{i=t}^{h}\mathcal{C}_{h+1-t}(0,0,n-i)Y_{n,h}(t,n-t,t)X_{n,h}(n-t,i,t).$$

Now, recall that $\mathcal{C}_{h+1-t}(0,0,n-i)=\sum_{l=1}^{n-i}\frac{1}{(-q)^l-1}$. This implies that it suffices to show that

$$Y_{n,h}(t,n-t,t)\sum_{i=t}^{h}(\sum_{l=1}^{n-i}\frac{1}{(-q)^l-1})X_{n,h}(n-t,i,t)=-\frac{(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1-(-q)^{-(h-t)}}. \tag{7.8}$$

Since $\sum_{i=t}^{h} X_{n,h}(n-t,i,t) = 0$ (by (7.1)), we have that

$$\sum_{i=t}^{h}\Big(\sum_{l=1}^{n-i}\frac{1}{(-q)^l-1}\Big)X_{n,h}(n-t,i,t) = \sum_{k=0}^{h-t-1}\frac{1}{(-q)^{n-t-k}-1}\sum_{i=t}^{t+k}X_{n,h}(n-t,i,t). \tag{7.9}$$

We claim that

$$\sum_{i=t}^{t+k}X_{n,h}(n-t,i,t) = \frac{(-1)^{t+k}(-q)^{\frac{(t+k+1)(t-k)}{2}}}{\prod_{l=1}^{h-t-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})(1-(-q)^{-(h-t)})}. \tag{7.10}$$

Indeed, it is easy to see that this holds for $k=0$. Now, assume that this holds for k. Then, we have that

$$\begin{aligned}
&\sum_{i=t}^{t+k+1}X_{n,h}(n-t,i,t) = \sum_{i=t}^{t+k+1}\frac{(-1)^i(-q)^{\frac{i}{2}-\frac{i^2}{2}+it}}{\prod_{l=1}^{h-i}(1-(-q)^{-l})\prod_{l=1}^{i-t}(1-(-q)^{-l})}\\
&= \frac{(-1)^{t+k}(-q)^{\frac{(t+k+1)(t-k)}{2}}}{\prod_{l=1}^{h-t-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})(1-(-q)^{-(h-t)})} + \frac{(-1)^{t+k+1}(-q)^{\frac{t+k+1}{2}-\frac{(t+k+1)^2}{2}+(t+k+1)t}}{\prod_{l=1}^{h-t-k-1}(1-(-q)^{-l})\prod_{l=1}^{k+1}(1-(-q)^{-l})}\\
&= \frac{(-1)^{t+k+1}(-q)^{\frac{(t+k+2)(t-k-1)}{2}}}{\prod_{l=1}^{h-t-k-2}(1-(-q)^{-l})\prod_{l=1}^{k+1}(1-(-q)^{-l})(1-(-q)^{-(h-t)})}.
\end{aligned}$$

This shows that (7.10) holds.

Now, by (7.10), we have that (7.9) can be written as

$$\frac{(-1)^t(-q)^{\frac{t^2}{2}+\frac{3t}{2}-n}}{(1-(-q)^{-(h-t)})}\times\sum_{k=0}^{h-t-1}\frac{(-1)^k(-q)^{-\frac{k(k-1)}{2}}}{(1-(-q)^{-(n-t-k)})\prod_{l=1}^{h-t-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})}. \tag{7.11}$$

In Lemma 7.7 below, we will prove that

$$\begin{aligned}
&\sum_{k=0}^{h-t-1}\frac{(-1)^k(-q)^{-\frac{k(k-1)}{2}}}{(1-(-q)^{-(n-t-k)})\prod_{l=1}^{h-t-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})}\\
&= \frac{(-1)^{h-t-1}(-q)^{-\frac{(h-t)(h-t-1)}{2}}(-q)^{-(n-h)(h-t-1)}}{\prod_{l=n-h+1}^{n-t}(1-(-q)^{-l})}.
\end{aligned} \tag{7.12}$$

Combining (7.11) and (7.12), we have that

$$Y_{n,h}(t,n-t,t)\sum_{i=t}^{h}\Big(\sum_{l=1}^{n-i}\frac{1}{(-q)^l-1}\Big)X_{n,h}(n-t,i,t) = \frac{-(-q)^{\frac{-(h-t)(h+t+1)}{2}}}{1-(-q)^{-(h-t)}}.$$

This finishes the proof of the proposition. □

Lemma 7.7. *For $0 \le t < h \le n$, we have*

$$\begin{aligned}
&\sum_{k=0}^{h-t-1}\frac{(-1)^k(-q)^{-\frac{k(k-1)}{2}}}{(1-(-q)^{-(n-t-k)})\prod_{l=1}^{h-t-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})}\\
&= \frac{(-1)^{h-t-1}(-q)^{-\frac{(h-t)(h-t-1)}{2}}(-q)^{-(n-h)(h-t-1)}}{\prod_{l=n-h+1}^{n-t}(1-(-q)^{-l})}.
\end{aligned}$$

Proof. For $N > 0$, we claim the following statement:

(7.13)
$$\sum_{k=0}^{N-1} \frac{(-1)^k (-q)^{-\frac{k(k-1)}{2}}}{(1-(-q)^{-(N-k)}X)\prod_{l=1}^{N-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})} = \frac{(-1)^{N-1}(-q)^{-\frac{N(N-1)}{2}}X^{N-1}}{\prod_{l=1}^{N}(1-(-q)^{-l}X)},$$

which specializes to the statement of the lemma when $X = (-q)^{-(n-h)}$. Therefore it suffices to show the claim (7.13).

Consider a Vandermonde matrix $\mathfrak{X} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{pmatrix}$, and let $\mathfrak{X}^{-1} = (y_{ij})_{1\le i,j\le N}$.

Note that y_{ij} is the X^{N-j}-coefficient of $\prod_{l=1,l\neq i}^{N} \frac{1-x_l X}{x_i - x_l}$.

Now, assume that $x_l = (-q)^{-l}$. Then, we have that

(7.14)
$$\sum_{j=1}^{N}\sum_{k=0}^{N-1} y_{(N-k)j}X^{N-j} = (-1)^{N-1}(-q)^{\frac{N(N-1)}{2}} \sum_{k=0}^{N-1} \frac{(-1)^k(-q)^{-\frac{k(k-1)}{2}}\prod_{l=1,l\neq N-k}^{N}(1-(-q)^{-l}X)}{\prod_{l=1}^{N-k-1}(1-(-q)^{-l})\prod_{l=1}^{k}(1-(-q)^{-l})}.$$

Now, note that

$$(1 \quad 0\dots 0)\mathfrak{X} = (1 \quad 1\dots 1) \Longleftrightarrow (1 \quad 0\dots 0) = (1 \quad 1\dots 1)\mathfrak{X}^{-1} = (\sum_{i=1}^{N} y_{i1} \quad \sum_{i=1}^{N} y_{i2} \dots \sum_{i=1}^{N} y_{iN}).$$

Hence, we have $\sum_{j=1}^{N}\sum_{k=0}^{N-1} y_{(N-k)j}X^{N-j} = X^{N-1}$. Combining this and (7.14), we have that (7.13) holds. This finishes the proof of the lemma. □

Now, let us state the following theorem.

Theorem 7.8. *Assume that* $0 \le h \le n$.

(1) Assume that $h+1 \le b \le n$, *and* $b+c = n$. *Then, we have* $D_{n,h}(0,b,c) = \frac{\prod_{l=h+1}^{b}(1-(-q)^l)}{(1-(-q)^{b-h})}$.

(2) Assume that $h-1 \le b \le n$, *and* $b+c = n-1$. *Then, we have*

$$D_{n,h}(1,b,c) = \begin{cases} 1 & \text{if } b = h-1, h, \\ \prod_{l=h+1}^{b}(1-(-q)^l) & \text{if } b \ge h+1. \end{cases}$$

Proof. Recall from Lemma 7.1 that for $a = 0$ or 1 (i.e., $b+c = n$ or $n-1$, respectively), we have

(7.15)
$$\begin{aligned} &D_{n,h}(a,b,c) = \textstyle\sum_{s=0}^{\min(h,b)} \sum_{i=s}^{\min(s+c,h)} M_{n,h}(a,b,c,i,s) \\ &= (-1)^{n+c+1}(-q)^{\frac{c}{2}+\frac{c^2}{2}-h^2-\frac{n}{2}-cn+hn+\frac{n^2}{2}-ch} \frac{\prod_{l=1}^{n-c-a}(1-(-q)^{-l})}{\prod_{l=1}^{n-h-c}(1-(-q)^{-l})} \\ &\quad \times \textstyle\sum_{s=0}^{\min(h,b)} \frac{(-1)^s(-q)^{\frac{s}{2}-\frac{s^2}{2}}\prod_{l=1}^{n-c-s+a-1}(1-(-q)^{-l})}{\prod_{l=1}^{s}(1-(-q)^{-l})\prod_{l=1}^{h-s}(1-(-q)^{-l})}. \end{aligned}$$

First, assume that $a=1$, $b=h-1$, and $c=n-h$. Then, we have

$$\sum_{s=0}^{\min(h,b)} \frac{(-1)^s(-q)^{\frac{s}{2}-\frac{s^2}{2}}\prod_{l=1}^{n-c-s+a-1}(1-(-q)^{-l})}{\prod_{l=1}^{s}(1-(-q)^{-l})\prod_{l=1}^{h-s}(1-(-q)^{-l})} = \sum_{s=0}^{h-1}\frac{(-1)^s(-q)^{\frac{s}{2}-\frac{s^2}{2}}}{\prod_{l=1}^{s}(1-(-q)^{-l})} = \frac{(-1)^{h-1}(-q)^{-\frac{h(h-1)}{2}}}{\prod_{l=1}^{h-1}(1-(-q)^{-l})}.$$

Therefore, we have $D_{n,h}(1,h-1,n-h)=1$.

Now, assume that $b\geq h$. In this case, we have that $\min(h,b)=h$. We claim that

$$\sum_{s=0}^{h}\frac{(-1)^s(-q)^{\frac{s}{2}-\frac{s^2}{2}}\prod_{l=1}^{N-s}(1-(-q)^{-l})}{\prod_{l=1}^{s}(1-(-q)^{-l})\prod_{l=1}^{h-s}(1-(-q)^{-l})} = \frac{(-1)^h(-q)^{-hN+\frac{h(h-1)}{2}}\prod_{l=1}^{N-h}(1-(-q)^{-l})}{\prod_{l=1}^{h}(1-(-q)^{-l})}, \tag{7.16}$$

where $N=n-c+a-1$.

To prove (7.16), we define the following constants: for $0\leq k\leq h$, $1\leq t\leq k+1$,

$$\omega_{k,t} = \begin{cases} \dfrac{(-1)^k(-q)^{-(t-1)N+\frac{(t-1)(t-2)}{2}-\frac{(k-t+1)(k-t+2)}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k-t+1}(1-(-q)^{-l})\prod_{l=1}^{h-k-1}(1-(-q)^{-l})\prod_{l=h-t+1}^{h}(1-(-q)^{-l})} & \text{if } k\leq h-1, \\ 0 & \text{if } k=h, t\neq h+1, \\ \dfrac{(-1)^h(-q)^{-hN+\frac{h(h-1)}{2}}\prod_{l=1}^{N-h}(1-(-q)^{-l})}{\prod_{l=1}^{h}(1-(-q)^{-l})} & \text{if } k=h, t=h+1, \end{cases}$$

$$\tau_{k,t} = \frac{(-1)^k(-q)^{-tN+\frac{t(t-1)}{2}-\frac{(k-t-1)(k-t)}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k-t}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})\prod_{l=h-t+1}^{h}(1-(-q)^{-l})}.$$

We claim the following two equations: for $k\geq 1$, $2\leq t\leq k$, we have

$$\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})} + \omega_{k-1,1} = \omega_{k,1}+\tau_{k,1}, \tag{7.17}$$

$$\tau_{k,t-1}+\omega_{k-1,t} = \omega_{k,t}+\tau_{k,t}. \tag{7.18}$$

First, we have

$$\begin{aligned} &\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})} + \omega_{k-1,1} \\ &= \frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k-1}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})}\Big\{\frac{(-q)^{-k}(1-(-q)^{-(h-k)})}{(1-(-q)^{-k})(1-(-q)^{-h})}+\frac{(-q)^{-(N-k+1)}}{1-(-q)^{-h}}\Big\} \\ &= \omega_{k,1}+\tau_{k,1}. \end{aligned}$$

This shows that (7.17) holds.

For (7.18), we have

$$\begin{aligned} \tau_{k,t-1}+\omega_{k-1,t} =& \frac{(-1)^k(-q)^{-(t-1)N+\frac{(t-1)(t-2)}{2}-\frac{(k-t)(k-t+1)}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k-t}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})\prod_{l=h-t+2}^{h}(1-(-q)^{-l})} \\ &\times\Big\{\frac{(-q)^{-(k-t+1)}(1-(-q)^{-(h-k)})}{(1-(-q)^{-(k-t+1)})(1-(-q)^{-(h-t+1)})}+\frac{(-q)^{-(N-k+1)}}{1-(-q)^{-(h-t+1)}}\Big\} \\ =&\omega_{k,t}+\tau_{k,t}. \end{aligned}$$

This shows that (7.18) holds. Now, by (7.17) and (7.18), we have that for $k \geq 1$,

$$\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})}+\sum_{t=1}^{k}\omega_{k-1,t}+\sum_{t=1}^{k-1}\tau_{k,t}=\sum_{t=1}^{k}\omega_{k,t}+\sum_{t=1}^{k}\tau_{k,t},$$

and hence

$$\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})}+\sum_{t=1}^{k}\omega_{k-1,t}=\sum_{t=1}^{k}\omega_{k,t}+\tau_{k,k}=\sum_{t=1}^{k+1}\omega_{k,t}. \tag{7.19}$$

Here, we used $\tau_{k,k}=\omega_{k,k+1}$ for the last identity.

Therefore, by (7.19), we have

$$\begin{aligned}
&\sum_{k=1}^{h}\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})}+\sum_{k=1}^{h}\sum_{t=1}^{k}\omega_{k-1,t}=\sum_{k=1}^{h}\sum_{t=1}^{k+1}\omega_{k,t}\\
\Longleftrightarrow&\sum_{k=1}^{h}\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})}+\omega_{0,1}=\sum_{t=1}^{h+1}\omega_{h,t}\\
\Longleftrightarrow&\sum_{k=0}^{h}\frac{(-1)^k(-q)^{\frac{k}{2}-\frac{k^2}{2}}\prod_{l=1}^{N-k}(1-(-q)^{-l})}{\prod_{l=1}^{k}(1-(-q)^{-l})\prod_{l=1}^{h-k}(1-(-q)^{-l})}=\omega_{h,h+1}\ \text{(since } \omega_{h,t}=0 \text{ for all } t<h+1)\\
&=\frac{(-1)^h(-q)^{-hN+\frac{h(h-1)}{2}}\prod_{l=1}^{N-h}(1-(-q)^{-l})}{\prod_{l=1}^{h}(1-(-q)^{-l})}.
\end{aligned}$$

This shows that (7.16) holds.

Combining (7.15) and (7.16), we have

$$\begin{aligned}
D_{n,h}(a,b,c)=(-1)^{n+c+h+1}(-q)^{\frac{c}{2}+\frac{c^2}{2}-h^2-\frac{n}{2}-cn+hn+\frac{n^2}{2}-ch-h(n-c+a-1)+\frac{h(h-1)}{2}}\\
\times\frac{\prod_{l=1}^{n-c-a}(1-(-q)^{-l})\prod_{l=1}^{n-c+a-1-h}(1-(-q)^{-l})}{\prod_{l=1}^{n-h-c}(1-(-q)^{-l})\prod_{l=1}^{h}(1-(-q)^{-l})}.
\end{aligned}$$

If $a=1$, then we have $b=n-c-1$, and $D_{n,h}(1,b,c)=\prod_{l=h+1}^{b}(1-(-q)^l)$. If $a=0$, then we have $b=n-c$, and $D_{n,h}(0,b,c)=\dfrac{\prod_{l=h+1}^{b}(1-(-q)^l)}{(1-(-q)^{b-h})}$. This finishes the proof of the theorem. □

8. Fourier transform

In this section, we will prove certain theorems on the Fourier transform of the analytic side of Conjecture 5.6. The main results of this section are Theorem 8.18, Theorem 8.19, Theorem 8.20, and Theorem 8.21. Recall that $\mathbb{V}$ is the space of special homomorphisms associated with $\mathcal{N}_n^{[h]}$ and it is split (resp. non-split) if h is odd (resp. even). For an integrable function f on $\mathbb{V}$, we denote by $\widehat{f}$ its Fourier transform $\widehat{f}(x):=\int_{\mathbb{V}}f(y)\psi(\mathrm{Tr}_{F/F_0}\langle x,y\rangle)dy, x\in\mathbb{V}$. For example, for an O_F-lattice $L\in\mathbb{V}$ of rank n, we have $\widehat{1}_L=\mathrm{vol}(L)1_{L^\vee}=q^{-\mathrm{val}(L)/2}1_{L^\vee}$, where we recall that $\mathrm{val}(L)$ is the sum of fundamental invariants of L.

Definition 8.1. Let $L\subset\mathbb{V}$ (resp. $L^\flat\subset\mathbb{V}$) be an O_F-lattice of rank n (resp. $n-1$).

(1) For simplicity, we set $D_{n,h}(L) \coloneqq \partial\mathrm{Pden}_{n,h}(L)$. Assume that $\lambda \in \mathcal{R}_n^{0+}$ be the fundamental invariants of L. Then by Corollary 6.7, we have $D_{n,h}(L) = D_{n,h}(\lambda)$, where $D_{n,h}(\lambda)$ is the Cho-Yamauchi constant defined in Definition 6.3.

(2) For $x \in \mathbb{V}\backslash L_F^\flat$, we define $\partial\mathrm{Den}_{L^\flat}^{n,h}(x) \coloneqq \sum_{L^\flat \subset L' \subset L'^\vee} D_{n,h}(L')1_{L'}(x)$, where $L' \subset \mathbb{V}$ are O_F-lattices of rank n.

(3) For $x \in \mathbb{V}\backslash L_F^\flat$, and $L'^\flat \supset L^\flat$, we define $\partial\mathrm{Den}_{L'^{\flat\circ}}^{n,h}(x) \coloneqq \sum_{L'^\flat \subset L' \subset L'^\vee, L' \cap L_F^\flat = L'^\flat} D_{n,h}(L')1_{L'}(x)$, where $L' \subset \mathbb{V}$ are O_F-lattices of rank n.

Then, we have that $\partial\mathrm{Den}_{L^\flat}^{n,h}(x) = \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \partial\mathrm{Den}_{L'^{\flat\circ}}^{n,h}(x)$.

By definition, we have that $\widehat{\partial\mathrm{Den}}_{L'^{\flat\circ}}^{n,h}(x) = \sum_{L'^\flat \subset L' \subset L'^\vee, L' \cap L_F^\flat = L'^\flat, x \in L'^\vee} D_{n,h}(L')\,\mathrm{vol}(L')$. Also, by [LZ22a, Lemma 7.2.1, Lemma 7.2.2, (7.4.2.3)], we have that

$$\begin{array}{ccc} [((\langle x\rangle + L'^\flat)^{\vee,\geq 0}/L'^\flat)\backslash(L_F^\flat/L'^\flat)] & \xrightarrow{\sim} & \{L'^\flat \subset L' \subset L'^\vee, L' \cap L_F^\flat = L'^\flat, x \in L'^\vee\} \\ u & \longmapsto & L'^\flat + \langle u\rangle. \end{array}$$

Now, assume that $x \perp L^\flat$ and $\mathrm{val}(\langle x, x\rangle) \leq -1$. Then, as in the proof of [LZ22a, Lemma 7.4.2], we have that

$$(\langle x\rangle + L'^\flat)^{\vee,\geq 0} = (L'^\flat)^{\vee,\geq 0} \oplus \langle x\rangle^\vee,$$

$$[((\langle x\rangle + L'^\flat)^{\vee,\geq 0}/L'^\flat)\backslash(L_F^\flat/L'^\flat)] \xrightarrow{\sim} (L'^\flat)^{\vee,\geq 0}/L'^\flat \times (\langle x\rangle^\vee\backslash\{0\})/O_F^\times.$$

Therefore, we have that

$$\begin{array}{ccc} (L'^\flat)^{\vee,\geq 0}/L'^\flat \times (\langle x\rangle^\vee\backslash\{0\})/O_F^\times & \xrightarrow{\sim} & \{L'^\flat \subset L' \subset L'^\vee, L' \cap L_F^\flat = L'^\flat, x \in L'^\vee\} \\ (u^\flat, u^\perp) & \longmapsto & L'^\flat + \langle u^\flat + u^\perp\rangle, \end{array}$$

(8.1)
$$\widehat{\partial\mathrm{Den}}_{L'^{\flat\circ}}^{n,h}(x) = \sum_{(u^\flat,u^\perp) \in (L'^\flat)^{\vee,\geq 0}/L'^\flat \times (\langle x\rangle^\vee\backslash\{0\})/O_F^\times} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle)\,\mathrm{vol}(L'^\flat + \langle u^\flat + u^\perp\rangle).$$

Now, we need to compute $D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle)$. First, let us define the following notations.

Definition 8.2. For $\lambda \in \mathcal{R}_{n-1}^{0+}$, we let $L_\lambda \subset \mathbb{V}$ be an O_F-lattice of rank $n-1$ with hermitian matrix A_λ. Consider a basis $\mathbb{B} = \{x_1, \ldots, x_{n-1}\}$ such that the hermitian matrix of L_λ with respect to $\mathbb{B}$ is A_λ.

(1) For $i \geq 0$, we define $L_{\lambda\geq i}$ to be the sublattice of L_λ generated by $\{x_1, \ldots, x_{t_{\geq i}(\lambda)}\}$. In particular, $L_{\lambda\geq 0} = L_\lambda$.

(2) For $i \geq 0$, we define $L_{\lambda=i}$ to be the sublattice of L_λ generated by x_j's such that $\mathrm{val}((x_j, x_j)) = i$. Therefore, the hermitian matrix of $L_{\lambda=i}$ is $\pi^i I_{t_i(\lambda)}$, where t_i is defined in Definition 6.11.

(3) For $i \geq j \geq 0$, we define $L_{(\lambda\geq i)-j}$ be an O_F-lattice of rank $t_{\geq i}(\lambda)$ with hermitian matrix $\mathrm{Diag}(\pi^{\lambda_1-j}, \pi^{\lambda_2-j}, \ldots, \pi^{\lambda_{t_{\geq i}(\lambda)}-j})$.

(4) For an O_F-lattice L, we define

$$\mu(L) = |(L^\vee)^{\geq 0}/L|,$$
$$\mu^+(L) = |(\pi L^\vee)^{\geq 1}/L|,$$
$$\mu^{++}(L) = |(\pi^2 L^\vee)^{\geq 2}/L|.$$

Consider a basis $\mathbb{B} = \{x_1, x_2, \ldots, x_a, x_{a+1}, \ldots, x_{a+b}, \ldots, x_{a+b+c}\}$ of $L'^\flat$ $(a+b+c = n-1)$ such that the hermitian matrix of L with respect to $\mathbb{B}$ is A_λ where $\lambda = (\lambda_1, \ldots, \lambda_a, 1^b, 0^c) \in \mathcal{R}^{0+}_{n-1}$, $\lambda_i \geq 2$. To simplify notation, we abbreviate $L_{\lambda\geq 2}$, $L_{\lambda=1}$ to $L'^\flat_2$, $L'^\flat_1$, respectively. In other words, $L'^\flat_2$ is the O_F-lattice generated by $\{x_1, \ldots, x_a\}$, and $L'^\flat_1$ is the O_F-lattice generated by $\{x_{a+1}, \ldots, x_{a+b}\}$.

Note that

$$(L'^\flat)^{\vee,\geq 0}/L'^\flat = (L'^\flat_2 \oplus L'^\flat_1)^{\vee,\geq 0}/(L'^\flat_2 \oplus L'^\flat_1).$$

Then, we have the following propositions.

Proposition 8.3. *Assume that* $\mathrm{val}((u^\perp, u^\perp)) \geq 2$*, then we have the followings.*

(1) (Case 1-1) If $u^\flat \in (\pi^2(L'^\flat_2)^\vee \oplus \pi(L'^\flat_1)^\vee)^{\geq 2}$, $D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a+1, b, c)$.

(2) (Case 1-2) If $u^\flat \in (\pi^2(L'^\flat_2)^\vee \oplus \pi(L'^\flat_1)^\vee)^{\geq 1} - (\pi^2(L'^\flat_2)^\vee \oplus \pi(L'^\flat_1)^\vee)^{\geq 2}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a, b+1, c).$$

(3) (Case 1-3) If $u^\flat \in (\pi^2(L'^\flat_2)^\vee \oplus \pi(L'^\flat_1)^\vee)^{\geq 0} - (\pi^2(L'^\flat_2)^\vee \oplus \pi(L'^\flat_1)^\vee)^{\geq 1}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a, b, c+1).$$

(4) (Case 2-1) If $u^\flat \in (\pi^2(L'^\flat_2)^\vee)^{\geq 0} \oplus ((L'^\flat_1)^\vee - \pi(L'^\flat_1)^\vee)^{\geq 0}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a+1, b-2, c+2).$$

(5) (Case 2-2) If $u^\flat \in ((\pi^2(L'^\flat_2)^\vee) \oplus ((L'^\flat_1)^\vee - \pi(L'^\flat_1)^\vee))^{\geq 0} - (\pi^2(L'^\flat_2)^\vee)^{\geq 0} \oplus ((L'^\flat_1)^\vee - \pi(L'^\flat_1)^\vee)^{\geq 0}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a, b-1, c+2).$$

(6) (Case 3-1) If $u^\flat \in ((\pi(L'^\flat_2)^\vee - \pi^2(L'^\flat_2)^\vee) \oplus (\pi(L'^\flat_1)^\vee))^{\geq 1}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a-1, b+2, c).$$

(7) (Case 3-2) If $u^\flat \in ((\pi(L'^\flat_2)^\vee - \pi^2(L'^\flat_2)^\vee) \oplus (\pi(L'^\flat_1)^\vee))^{\geq 0} - ((\pi(L'^\flat_2)^\vee - \pi^2(L'^\flat_2)^\vee) \oplus (\pi(L'^\flat_1)^\vee))^{\geq 1}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a, b, c+1).$$

(8) (Case 4-1) If $u^\flat \in (\pi(L'^\flat_2)^\vee - \pi^2(L'^\flat_2)^\vee)^{\geq 0} \oplus ((L'^\flat_1)^\vee - \pi(L'^\flat_1)^\vee)^{\geq 0}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a+1, b-2, c+2).$$

(9) (Case 4-2) If

$$u^\flat \in ((\pi(L'^\flat_2)^\vee - \pi^2(L'^\flat_2)^\vee) \oplus ((L'^\flat_1)^\vee - \pi(L'^\flat_1)^\vee))^{\geq 0} - (\pi(L'^\flat_2)^\vee - \pi^2(L'^\flat_2)^\vee)^{\geq 0} \oplus ((L'^\flat_1)^\vee - \pi(L'^\flat_1)^\vee)^{\geq 0},$$

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a, b-1, c+2).$$

(10) (Case 5) If $u^\flat \in (((L'^\flat_2)^\vee - \pi(L'^\flat_2)^\vee) \oplus (L'^\flat_1)^\vee)^{\geq 0}$, $D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = D_{n,h}(a-1, b, c+2)$.

Proof. This is a linear algebra problem, so we skip the proof of the proposition. □

Proposition 8.4. *Assume that* $\mathrm{val}((u^\perp, u^\perp)) = 1$*, then we have the followings.*

(1) *(Case 1-1) If* $u^\flat \in (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 2}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp \rangle) = D_{n,h}(a, b+1, c).$$

(2) *(Case 1-2-1) If* $u^\flat \in (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 1} - (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 2}$, *and* $\mathrm{val}(\langle u^\flat + u^\perp, u^\flat + u^\perp \rangle) = 1$

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp \rangle) = D_{n,h}(a, b+1, c).$$

(3) *(Case 1-2-2) If* $u^\flat \in (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 1} - (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 2}$, *and* $\mathrm{val}(\langle u^\flat + u^\perp, u^\flat + u^\perp \rangle) \geq 2$, *we have*

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp \rangle) = D_{n,h}(a+1, b, c).$$

(4) *(Case 1-3) If* $u^\flat \in (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 0} - (\pi^2(L_2^\flat)^\vee \oplus \pi(L_1^\flat)^\vee)^{\geq 1}$,

$$D_{n,h}(L'^\flat + \langle u^\flat + u^\perp \rangle) = D_{n,h}(a, b, c+1).$$

The other cases: (Case 2-1), (Case 2-2), (Case 3-1), (Case 3-2), (Case 4-1), (Case 4-2), and (Case 5) are the same as Proposition 8.3.

Proof. This is a linear algebra problem, so we skip the proof of the proposition. □

Remark 8.5. We note that the only difference between the case $\mathrm{val}((u^\perp, u^\perp)) \geq 2$ (Proposition 8.3) and the case $\mathrm{val}((u^\perp, u^\perp)) = 1$ (Proposition 8.4) is the cases 1-1, 1-2, and 1-3.

Proposition 8.6. *Assume that* $\lambda \in \mathcal{R}_n^{0+}$ *and* L_λ *is an* O_F*-lattice with hermitian matrix* A_λ.

(1) *If* $\lambda \geq (1, \ldots, 1)$, *we have*

$$\mu^+(L_\lambda) = \mu(L_{\lambda - 1}).$$

(2) *If* $\lambda \geq (2, \ldots, 2)$, *we have*

$$\mu^{++}(L_\lambda) = \mu^+(L_{\lambda - 1}).$$

Proof. By definition, we have that

$$\mu(L_{\lambda-1}) = |(L_{\lambda-1}^\vee)^{\geq 0}/L_{\lambda-1}|,$$
$$\mu^+(L_\lambda) = |(\pi L_\lambda^\vee)^{\geq 1}/L_\lambda|.$$

Note that the fundamental invariants of $L_{\lambda-1}^\vee$ (resp. $\pi L_\lambda^\vee$)) are $(-\lambda_1 + 1, \ldots, -\lambda_n + 1)$ (resp. $(-\lambda_1 + 2, \ldots, -\lambda_n + 2)$). Now, for $(L_{\lambda-1}^\vee)^{\geq 0}/L_{\lambda-1}$, and the hermitian form $\langle \cdot, \cdot \rangle$, we consider the same set with the hermitian form $\pi\langle \cdot, \cdot \rangle$. Then, this is isomorphic to $(\pi L_\lambda^\vee)^{\geq 1}/L_\lambda$. This proves (1). The proof of (2) is similar. □

Proposition 8.7. *For* $\lambda \in \mathcal{R}_{n-1}^{0+}$ *and* $L_\lambda = L_{\lambda \geq 2} \oplus L_{\lambda = 1} =: L_2 \oplus L_1$, *we have the followings.*

(1) *(Case 1-1) We have that*

$$|(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 2}/L_2 \oplus L_1| = \mu(L_{(\lambda \geq 2)-2}).$$

(2) *(Case 1-2) We have that*

$$|\{(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 1} - (\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 2}\}/L_2 \oplus L_1|$$
$$= q^{2t_{\geq 3}(\lambda)} \mu(L_{(\lambda \geq 3)-3}) - \mu(L_{(\lambda \geq 2)-2}).$$

(3) (Case 1-3+Case 3-2) We have that

$$\begin{aligned}&|\{(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 0} - (\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 1}\}/L_2 \oplus L_1| \\ &+|\{((\pi(L_2)^\vee - \pi^2(L_{\lambda\geq 2})^\vee) \oplus (\pi(L_1)^\vee))^{\geq 0} - ((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus (\pi(L_1)^\vee))^{\geq 1}\}/L_2 \oplus L_1| \\ &= q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2}) - \mu(L_{(\lambda\geq 2)-1}).\end{aligned}$$

(4) (Case 2-1+Case 4-1) We have that

$$\begin{aligned}&|\{(\pi^2(L_2^\flat)^\vee)^{\geq 0} \oplus ((L_1^\flat)^\vee - \pi(L_1^\flat)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ &+|\{(\pi(L_2^\flat)^\vee - \pi^2(L_2^\flat)^\vee)^{\geq 0} \oplus ((L_1^\flat)^\vee - \pi(L_1^\flat)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ &= q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2}) \times (\mu(L_{\lambda=1}) - 1).\end{aligned}$$

(5) (Case 2-2+Case 4-2) We have that

$$\begin{aligned}&|\{((\pi^2(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0} - (\pi^2(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ &+|\{((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus ((L_1)^\vee - \pi(L_1)^\vee))^{\geq 0}\}/L_2 \oplus L_1| \\ &-|\{(\pi(L_2)^\vee - \pi^2(L_2)^\vee)^{\geq 0} \oplus ((L_1)^\vee - \pi(L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| \\ &= q^{2t_{\geq 2}(\lambda)}\{\mu(L_{(\lambda\geq 2)-2} \oplus L_{\lambda=1}) - \mu(L_{(\lambda\geq 2)-2})\mu(L_{\lambda=1})\}.\end{aligned}$$

(6) (Case 3-1) We have that

$$|\{((\pi(L_2)^\vee - \pi^2(L_2)^\vee) \oplus (\pi(L_1)^\vee))^{\geq 1}\}/L_2 \oplus L_1| = \mu(L_{(\lambda\geq 2)-1}) - q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3}).$$

(7) (Case 5) We have that

$$|\{(((L_2)^\vee - \pi(L_2)^\vee) \oplus (L_1)^\vee)^{\geq 0}\}/L_2 \oplus L_1| = \mu(L_{\lambda\geq 2} \oplus L_{\lambda=1}) - q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2} \oplus L_{\lambda=1}).$$

Here, we choose the following convention: $\forall i \geq j \geq 0$, if $L_{\lambda\geq i}$ is empty, then $\mu(L_{(\lambda\geq i)-j}) = 1$.

Proof. We only prove the case (Case 1-2). The other cases can be proved similarly.

In (Case 1-2), note that $\pi(L_1)^\vee = L_1$. Therefore, we have

$$|\{(\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 1} - (\pi^2(L_2)^\vee \oplus \pi(L_1)^\vee)^{\geq 2}\}/L_2 \oplus L_1| = |\{(\pi^2(L_2)^\vee)^{\geq 1} - (\pi^2(L_2)^\vee)^{\geq 2}\}/L_2|.$$

If $L_{\lambda\geq 3}$ is empty, then we have that $\pi^2 L_2^\vee = L_2 = L_{\lambda=2}$. Therefore,

$$|\{(\pi^2(L_2)^\vee)^{\geq 1}\}/L_2| = |\{(\pi^2(L_2)^\vee)^{\geq 2}\}/L_2| = 1.$$

If $L_{\lambda\geq 3}$ is not empty, then we have

$$\begin{aligned}&|\{(\pi^2(L_2)^\vee)^{\geq 1}\}/L_2| = |\{(\pi^2 L_{\lambda\geq 3}^\vee)^{\geq 1}\}/L_{\lambda\geq 3}| = q^{2t_{\geq 3}(\lambda)}|\{(\pi(\pi^{-1}L_{\lambda\geq 3})^\vee)^{\geq 1}\}/\pi^{-1}L_{\lambda\geq 3}| \\ &= q^{2t_{\geq 3}(\lambda)}\mu^+(\pi^{-1}L_{\lambda\geq 3}) = q^{2t_{\geq 3}(\lambda)}\mu^+(L_{(\lambda\geq 3)-2}) = q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3}).\end{aligned}$$

Here, we used Proposition 8.6. Also, by definition, we have that $|\{(\pi^2(L_2)^\vee)^{\geq 2}\}/L_2| = \mu^{++}(L_2) = \mu(L_{(\lambda\geq 2)-2})$. This finishes the proof of the proposition in the case (Case 1-2). □

Proposition 8.8. *(cf.* [LZ22b, Lemma 8.2.3], [HLSY23, Lemma 8.4]*) Assume that L (resp. M) is an O_F-lattice of rank n with hermitian matrix A_λ (resp. A_η) such that $\lambda, \eta \in \mathcal{R}_n^{0+}$, $\lambda, \eta \geq (1, \ldots, 1)$. Assume further that $L \subset M \subset \pi^{-1}L$. Then, we have*

$$|((L^\vee)^{\geq 0} \backslash (M^\vee)^{\geq 0})/L| = q^{2n-1}|((\pi L^\vee)^{\geq 1} \backslash (\pi M^\vee)^{\geq 1})/L|.$$

Proof. Since the proof follows the argument in [LZ22b, Lemma 8.2.3], we omit the details here. □

Proposition 8.9. *(cf.* [LZ22b, Lemma 8.2.6], [HLSY23, Lemma 8.6]*) Assume that L is an O_F-lattice of rank n and $e = \{e_1, \dots, e_n\}$ is a basis of L. Also, assume that the hermitian matrix of L with respect to e is A_λ where $\lambda \in \mathcal{R}_n^{0+}$, $\lambda \geq (1, \dots, 1)$. We choose an O_F-lattice M as follows.*

(1) If $\lambda_1 \geq 3$, we choose an O_F-lattice $M \supset L$ such that $M = O_F(\frac{1}{\pi}e_1) \oplus \oplus_{i \neq 1} O_F(e_i)$ with fundamental invariants $(\lambda_1 - 2, \lambda_2, \dots, \lambda_n)$.

(2) If $\lambda_1 = \lambda_2 = 2$ (so, $\lambda_i \leq 2$ for all i), we choose an O_F-lattice $M \supset L$ such that the fundamental invariants of M are $(\lambda_1 - 1, \lambda_2 - 1, \lambda_3, \dots, \lambda_n)$.

Then, we have that

$$|((\pi L^\vee)^{\geq 0} \backslash (\pi M^\vee)^{\geq 0})/L| = q|((\pi L^\vee)^{\geq 1} \backslash (\pi M^\vee)^{\geq 1})/L|.$$

Proof. Since the proof follows the argument in [LZ22b, Lemma 8.2.6], we omit the details here. □

Proposition 8.10. *Assume that $\lambda \in \mathcal{R}_n^{0+}$.*

(1) If $\lambda_1 \geq 3$, we define $\eta = (\lambda_1 - 2, \lambda_2, \dots, \lambda_n) \in \mathcal{R}_n^{0+}$, (if necessary, we change the order of η_i's so that $\eta \in \mathcal{R}_n^{0+}$). Then, we have

$$\mu(L_\lambda) = \mu(L_{\lambda \geq 1}) \quad = q^2 \mu(L_{\eta \geq 1}) + q^{2t_{\geq 1}(\lambda) + 2t_{\geq 2}(\lambda) - 2} \mu(L_{(\lambda \geq 2)-2}) - q^{2t_{\geq 1}(\lambda) + 2t_{\geq 2}(\lambda)} \mu(L_{(\eta \geq 2)-2}).$$

(2) If $\lambda_1 = \lambda_2 = 2$, we define $\eta = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_n) \in \mathcal{R}_n^{0+}$, (if necessary, we change the order of η_i's so that $\eta \in \mathcal{R}_n^{0+}$). Then, we have

$$\mu(L_\lambda) = \mu(L_{\lambda \geq 1}) \quad = q^2 \mu(L_{\eta \geq 1}) + q^{2t_{\geq 1}(\lambda) + 2t_{\geq 2}(\lambda) - 2} \mu(L_{(\lambda \geq 2)-2}) - q^{2t_{\geq 1}(\eta) + 2t_{\geq 2}(\eta)} \mu(L_{(\eta \geq 2)-2}).$$

Here, we choose the following convention: If $L_{\lambda \geq 2}$ is empty, we assume $\mu(L_{(\lambda \geq 2)-2}) = 1$.

Proof. By Proposition 8.8 and Proposition 8.9, we have

$$|((L_\lambda^\vee)^{\geq 0} \backslash (L_\eta^\vee)^{\geq 0})/L_\lambda| = q^{2t_{\geq 1}(\lambda) - 1} |((\pi L_\lambda^\vee)^{\geq 1} \backslash (\pi L_\eta^\vee)^{\geq 1})/L_\lambda|,$$

$$|((\pi L_\lambda^\vee)^{\geq 0} \backslash (\pi L_\eta^\vee)^{\geq 0})/L_\lambda| = q|((\pi L_\lambda^\vee)^{\geq 1} \backslash (\pi L_\eta^\vee)^{\geq 1})/L_\lambda|.$$

Therefore, we have

$$|((L_\lambda^\vee)^{\geq 0} \backslash (L_\eta^\vee)^{\geq 0})/L_\lambda| = q^{2t_{\geq 1}(\lambda) - 2} |((\pi L_\lambda^\vee)^{\geq 0} \backslash (\pi L_\eta^\vee)^{\geq 0})/L_\lambda|. \tag{8.2}$$

Note that $|(L_\lambda^\vee)^{\geq 0}/L_\lambda| = \mu(L_\lambda)$, and $|(L_\eta^\vee)^{\geq 0}/L_\lambda| = q^2 |(L_\eta^\vee)^{\geq 0}/L_\eta| = q^2 \mu(L_\eta)$.

Also, note that $(\pi L_\lambda^\vee)^{\geq 0}/L_\lambda = (\pi (L_{\lambda \geq 2})^\vee)^{\geq 0}/L_{\lambda \geq 2}$, and it is trivial if $L_{\lambda \geq 2}$ is empty. Since the fundamental invariants of $L_{\lambda \geq 2}$ are at least 2, any element x of $\pi^{-1} L_{\lambda \geq 2}$ has valuation at least 0. Therefore, we have

$$\begin{aligned} &|(\pi L_\lambda^\vee)^{\geq 0}/L_\lambda| = |(\pi (L_{\lambda \geq 2})^\vee)^{\geq 0}/L_{\lambda \geq 2}| = q^{2t_{\geq 2}(\lambda)} |((\pi^{-1} L_{\lambda \geq 2})^\vee)^{\geq 0}/(\pi^{-1} L_{\lambda \geq 2})| \\ &= q^{2t_{\geq 2}(\lambda)} \mu(\pi^{-1} L_{\lambda \geq 2}) = q^{2t_{\geq 2}(\lambda)} \mu(L_{(\lambda \geq 2)-2}). \end{aligned}$$

Similarly, we have $|(\pi L_\eta^\vee)^{\geq 0}/L_\lambda| = q^2 |(\pi L_\eta^\vee)^{\geq 0}/L_\eta| = q^{2+2t_{\geq 2}(\eta)} \mu(L_{(\eta \geq 2)-2})$.

Now, by (8.2) and the fact that $t_{\geq 1}(\lambda) = t_{\geq 1}(\eta)$, we have the proof of the proposition. □

Lemma 8.11. *For $\lambda \in \mathcal{R}_n^{0+}$, $\lambda \geq (1, 1, \dots, 1)$, we have that*

$$\mu(L_\lambda) = q^{2n-1} \mu(L_{\lambda - 1}) - (-q)^{|\lambda| - 1}(q - 1).$$

Proof. First, consider the case $\lambda = (1, 1, \dots, 1)$. Then, it is easy to see that

$$\begin{aligned}\mu(L_{1,\dots,1}) &= |\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_1^{q+1} + \cdots + x_n^{q+1} = 0\}| \\ &= q^{2n-1} + (-q)^n + (-q)^{n-1} = q^{2n-1}\mu(L_{0,\dots,0}) - (-q)^{n-1}(q-1).\end{aligned}$$

Similarly, if $\lambda = (2, \overbrace{1, \dots, 1}^{n-1})$, we have that

$$\mu(L_\lambda) = |((L_{2,1,\dots,1})^\vee)^{\geq 0}/L_{2,1,\dots,1}| = q^2\mu(L_{1,\dots,1}) = q^{2n-1}\mu(L_1) - (-q)^n(q-1).$$

Now, we will prove the lemma by induction on $|\lambda|$. We distinguish the cases $\lambda = (\overbrace{2, \dots, 2}^{a}, \overbrace{1, \dots, 1}^{b})$, $\lambda = (\overbrace{3, \dots, 3}^{n-a-b}, \overbrace{2, \dots, 2}^{a}, \overbrace{1, \dots, 1}^{b})$, $\lambda = (\lambda_1 = 4, \lambda_2, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^{a}, \overbrace{1, \dots, 1}^{b})$, $\lambda_{n-a-b} \geq 3$, and $\lambda = (\lambda_1, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^{a}, \overbrace{1, \dots, 1}^{b})$, $\lambda_1 \geq 5$. For simplicity, we present the proof for the fourth case, as the other cases are similar. Assume that $\lambda = (\lambda_1, \dots, \lambda_{n-a-b}, \overbrace{2, \dots, 2}^{a}, \overbrace{1, \dots, 1}^{b})$, $\lambda_1 \geq 5$. In this case, let $\eta = (\lambda_1 - 2, \lambda_2, \dots, \lambda_n)$ (if necessary, we change the order of η_i's so that $\eta \in \mathcal{R}_n^{0+}$). Then, by Proposition 8.10 (1), we have

$$\mu(L_\lambda) = q^2\mu(L_\eta) + q^{4n-2b-2}\mu(L_{(\lambda \geq 2)-2}) - q^{4n-2b}\mu(L_{(\eta \geq 2)-2}),$$

$$\mu(L_{\lambda-1}) = q^2\mu(L_{\eta-1}) + q^{4n-2a-4b-2}\mu(L_{(\lambda \geq 3)-3}) - q^{4n-2a-4b}\mu(L_{(\eta \geq 3)-3}).$$

Therefore,

$$\begin{aligned}\mu(L_\lambda) - q^{2n-1}\mu(L_{\lambda-1}) &= q^2\{\mu(L_\eta) - q^{2n-1}\mu(L_{\eta-1})\} \\ &\quad + q^{4n-2b-2}\{\mu(L_{(\lambda \geq 2)-2}) - q^{2n-2a-2b-1}\mu(L_{(\lambda \geq 3)-3})\} \\ &\quad - q^{4n-2b}\{\mu(L_{(\eta \geq 2)-2}) - q^{2n-2a-2b-1}\mu(L_{(\eta \geq 3)-3})\} \\ \text{(by our inductive hypothesis)} \quad &= -(-q)^{|\eta|+1}(q-1) + q^{4n-2b-2}(-(-q)^{|\lambda|-b-2(n-b)-1}(q-1)) \\ &\quad - (-q)^{4n-2b}(-(-q)^{|\eta|-b-2(n-b)-1}(q-1)) \\ &= -(-q)^{|\lambda|-1}(q-1).\end{aligned}$$

This finishes the proof of the lemma. □

Lemma 8.12. *Assume that $a \geq 1$, and $j \in \mathbb{Z}$. Let $\kappa_{a,i}$ be the constants such that*

$$(1-X)(1-(-q)X)\dots(1-(-q)^{a-2}X) = \sum_{i=0}^{a-1} \kappa_{a,i}X^i.$$

Then, we have

$$D_{n,h}(a,b,c) = \sum_{i=0}^{a-1} \kappa_{a,i}(-q)^{i(a+b-h+1)} D_{n-i,h-i}(1, b+a-1-i, c).$$

Proof. This follows from Theorem 7.4. □

Definition 8.13. Assume that $a + b + c = n - 1$.

(1) We define the function $E_{n,h}(a,b,c)$ by

$$E_{n,h}(a,b,c) := \begin{array}{rcl} & & D_{n,h}(a+1,b,c) \\ +((-q)^a-1) & \times & D_{n,h}(a,b+1,c) \\ +(-q)^a((-q)^a-1) & \times & D_{n,h}(a,b,c+1) \\ -(-q)^{2a}(1-(-q)^b)(1-(-q)^{b-1}) & \times & D_{n,h}(a+1,b-2,c+2) \\ -(-q)^{2a+b-1}(1-(-q)^b)(1-(-q)^a) & \times & D_{n,h}(a,b-1,c+2). \end{array}$$

(2) We define the function $F_{n,h}(a,b,c)$ by

$$F_{n,h}(a,b,c) := \begin{array}{rcl} & & D_{n,h}(a,b+1,c) \\ +((-q)^{a-1}-1) & \times & D_{n,h}(a-1,b+2,c) \\ -(-q)^{a-1} & \times & D_{n,h}(a,b,c+1) \\ +(-q)^{2a+b-1}(1-(-q)^b) & \times & D_{n,h}(a,b-1,c+2) \\ +(-q)^{2a+2b-1}(1-(-q)^{a-1}) & \times & D_{n,h}(a-1,b,c+2) \end{array}$$

Lemma 8.14. *Assume that $a \geq 1$. Then, we have $E_{n,h}(a,b,c) = 0$.*

Proof. To simplify notation, we use the following convention: for $k \geq 1$,

$$\begin{array}{ll} X_0^{[k]} := D_{n-k+1,h-k+1}(1,b+a-k+1,c), & X_0^{[1]} = X_0, \\ X_1^{[k]} := D_{n-k+1,h-k+1}(1,b+a-k,c+1), & X_1^{[1]} = X_1, \\ X_2^{[k]} := D_{n-k+1,h-k+1}(1,b+a-k-1,c+2), & X_2^{[1]} = X_2. \end{array}$$

Also, we use the convention $X_i^{[j]}X_i^{[k]} = X_i^{[j+k]}$, which should not be interpreted as actual multiplication. Then, by Lemma 8.12, we have that

$$\begin{aligned} &D_{n,h}(a+1,b,c) = X_0(1-(-q)^{a+b-h+2}X_0)\dots(1-(-q)^{2a+b-h+1}X_0), \\ &D_{n,h}(a,b+1,c) = X_0(1-(-q)^{a+b-h+2}X_0)\dots(1-(-q)^{2a+b-h}X_0), \\ &D_{n,h}(a,b,c+1) = X_1(1-(-q)^{a+b-h+1}X_1)\dots(1-(-q)^{2a+b-h-1}X_1), \\ &D_{n,h}(a+1,b-2,c+2) = X_2(1-(-q)^{a+b-h}X_2)\dots(1-(-q)^{2a+b-h-1}X_2), \\ &D_{n,h}(a,b-1,c+2) = X_2(1-(-q)^{a+b-h}X_2)\dots(1-(-q)^{2a+b-h-2}X_2). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} &D_{n,h}(a+1,b,c) + ((-q)^a-1)D_{n,h}(a,b+1,c) \\ &= (-q)^a X_0(1-(-q)^{a+b-h+1}X_0)\dots(1-(-q)^{2a+b-h}X_0) \\ &= (-q)^a X_0(1-(-q)^{2a+b-h}X_0)(\textstyle\sum_{t=0}^{a-1}\kappa_{a,t}(-q)^{t(a+b-h+1)}X_0^{[t]}). \end{aligned}$$

This implies that for $1 \leq t \leq a+1$, $X_0^{[t]}$-terms are

(8.3)
$$\begin{cases} (-q)^a\{\kappa_{a,t-1}(-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h}\kappa_{a,t-2}(-q)^{(a+b-h+1)(t-2)}\}X_0^{[t]} & \text{if } t \neq a+1, \\ -(-q)^{3a+b-h}\kappa_{a,a-1}(-q)^{(a+b-h+1)(a-1)}X_0^{[a+1]} & \text{if } t = a+1. \end{cases}$$

Similarly, for $1 \leq t \leq a$, $X_1^{[t]}$-terms in $(-q)^a((-q)^a-1)D_{n,h}(a,b,c+1)$ are

(8.4)
$$(-q)^a((-q)^a-1)\kappa_{a,t-1}(-q)^{(a+b-h+1)(t-1)}X_1^{[t]}.$$

Finally, for $1 \le t \le a+1$, $X_2^{[t]}$-terms in $-(-q)^{2a}(1-(-q)^b)(1-(-q)^{b-1})D_{n,h}(a+1,b-2,c+2) - (-q)^{2a+b-1}(1-(-q)^b)(1-(-q)^a)D_{n,h}(a,b-1,c+2)$ are

$$\begin{cases} \{-\kappa_{a,t-1}(-q)^{(a+b-h)(t-1)+2a}(1-(-q)^b)(1-(-q)^{a+b-1}) \\ +\kappa_{a,t-2}(-q)^{(a+b-h)(t-2)+4a+b-h-1}(1-(-q)^b)(1-(-q)^{b-1})\}X_2^{[t]} & \text{if } t \neq a+1, \\ \kappa_{a,a-1}(-q)^{(a+b-h)(a-1)+4a+b-h-1}(1-(-q)^b)(1-(-q)^{b-1})X_2^{[a+1]} & \text{if } t = a+1. \end{cases} \tag{8.5}$$

Note that if $a+b < h-1$, then $X_0^{[t]}, X_1^{[t]}$, and $X_2^{[t]}$ are all zero. Therefore, all of the degree t-terms of $E_{n,h}(a,b,c)$ are 0, and hence $E_{n,h}(a,b,c) = 0$ in this case.

Now, assume that $a+b \ge h+2$. Then, by Theorem 7.8, we have that

$$X_0^{[t]} = \textstyle\prod_{l=h-t+2}^{a+b-t+1}(1-(-q)^l), X_1^{[t]} = \prod_{l=h-t+2}^{a+b-t}(1-(-q)^l), X_2^{[t]} = \prod_{l=h-t+2}^{a+b-t-1}(1-(-q)^l). \tag{8.6}$$

Combining (8.3), (8.4), (8.5), and (8.6), we can see that the degree t-terms of $E_{n,h}(a,b,c)$ can be written as follows: if $1 \le t \le a$, we have

$$\begin{aligned} &\prod_{l=h-t+2}^{a+b-t-1}(1-(-q)^l)\Big\{(-q)^a\{\kappa_{a,t-1}(-q)^{(a+b-h+1)(t-1)} - (-q)^{2a+b-h}\kappa_{a,t-2}(-q)^{(a+b-h+1)(t-2)}\} \\ &\quad \times (1-(-q)^{a+b-t+1})(1-(-q)^{a+b-t}) \\ &\quad + (-q)^a((-q)^a-1)\kappa_{a,t-1}(-q)^{(a+b-h+1)(t-1)}(1-(-q)^{a+b-t}) \\ &\quad + \{-\kappa_{a,t-1}(-q)^{(a+b-h)(t-1)+2a}(1-(-q)^b)(1-(-q)^{a+b-1}) \\ &\quad + \kappa_{a,t-2}(-q)^{(a+b-h)(t-2)+4a+b-h-1}(1-(-q)^b)(1-(-q)^{b-1})\}\Big\} \\ &\quad = ((-q)^{a+2b}-(-q)^t)(-q)^{(a+b-h)(t-2)+3a+b-h-t-2} \\ &\quad \{-((-q)^{t+1}-(-q)^2)\kappa_{a,t-1} + ((-q)^t-(-q)^{a+1})\kappa_{a,t-2}\}. \end{aligned} \tag{8.7}$$

Now, we claim that

$$\{-((-q)^{t+1}-(-q)^2)\kappa_{a,t-1} + ((-q)^t-(-q)^{a+1})\kappa_{a,t-2}\} = 0. \tag{8.8}$$

Recall from Lemma 8.12 that $(1-X)(1-(-q)X)\dots(1-(-q)^{a-2}X) = \sum_{t=0}^{a-1}\kappa_{a,t}X^t$, and $(1-(-q)X)(1-(-q)^2X)\dots(1-(-q)^{a-1}X) = \sum_{t=0}^{a-1}\kappa_{a,t}(-q)^tX^t$. Therefore,

$$(1-(-q)^{a-1}X)\sum_{t=0}^{a-1}\kappa_{a,t}X^t = (1-X)\sum_{t=0}^{a-1}\kappa_{a,t}(-q)^tX^t.$$

By comparing degree $t-1$ terms, we have that

$$-((-q)^{t-1}-1)\kappa_{a,t-1} + ((-q)^{t-2}-(-q)^{a-1})\kappa_{a,t-2} = 0. \tag{8.9}$$

Therefore, we have that (8.8) holds, and hence (8.7) is zero. This shows that the degree t-term of $E_{n,h}(a,b,c)$ is zero for $1 \le t \le a$.

For $t = a+1$, we have that the degree $a+1$-terms of $E_{n,h}(a,b,c)$ are

$$\begin{aligned} &\textstyle\prod_{l=h-a+1}^{b-2}(1-(-q)^l) \quad \Big\{ -(-q)^{3a+b-h}\kappa_{a,a-1}(-q)^{(a+b-h+1)(a-1)}(1-(-q)^b)(1-(-q)^{b-1}) \\ &\qquad +\kappa_{a,a-1}(-q)^{(a+b-h)(a-1)+4a+b-h-1}(1-(-q)^b)(1-(-q)^{b-1})\Big\} = 0. \end{aligned}$$

This shows that $E_{n,h}(a,b,c) = 0$ holds when $a+b \ge h+2$.

Now, the remaining cases are $a+b=h-1$, h, and $h+1$.

When $a+b=h-1$, we have that $X_0^{[t]}=1$, $X_1^{[t]}=X_2^{[t]}=0$ by Theorem 7.8. Therefore, we have that $D_{n,h}(a+1,b,c)+((-q)^a-1)D_{n,h}(a,b+1,c)=(-q)^a(1-(-q)^{a+b-h+1})\dots(1-(-q)^{2a+b-h})=0$ (since $a+b-h+1=0$). Also, we have $D_{n,h}(a,b,c+1)=0$, $D_{n,h}(a+1,b-2,c+2)=0$, and $D_{n,h}(a,b-1,c+2)=0$. This shows that $E_{n,h}(a,b,c)=0$ holds when $a+b=h-1$.

The cases $a+b=h$ and $a+b=h-1$ are similar. This finishes the proof of the lemma. □

Lemma 8.15. *Assume that $a=0$. Then, we have*

$$E_{n,h}(a,b,c)=\begin{cases} 0 & \text{if } b\geq h+2 \text{ or } b\leq h-2,\\ 1 & \text{if } b=h-1,h,\\ q^{2h+1}+(-q)^h & \text{if } b=h+1.\end{cases}$$

Proof. This follows from Theorem 7.8. □

Lemma 8.16. *Assume that $a\geq 2$. Then, we have*

$$F_{n,h}(a,b,c)=0$$

Proof. Here we follow the notation in the proof of Lemma 8.14. Then, by Lemma 8.12, we have

$$\begin{aligned}
&D_{n,h}(a,b+1,c)=X_0(1-(-q)^{a+b-h+2}X_0)\dots(1-(-q)^{2a+b-h}X_0),\\
&D_{n,h}(a-1,b+2,c)=X_0(1-(-q)^{a+b-h+2}X_0)\dots(1-(-q)^{2a+b-h-1}X_0),\\
&D_{n,h}(a,b,c+1)=X_1(1-(-q)^{a+b-h+1}X_1)\dots(1-(-q)^{2a+b-h-1}X_1),\\
&D_{n,h}(a,b-1,c+2)=X_2(1-(-q)^{a+b-h}X_2)\dots(1-(-q)^{2a+b-h-2}X_2),\\
&D_{n,h}(a-1,b,c+2)=X_2(1-(-q)^{a+b-h}X_2)\dots(1-(-q)^{2a+b-h-3}X_2).
\end{aligned}$$

Therefore, for $1\leq t\leq a$, $X_0^{[t]}$-terms of $D_{n,h}(a,b+1,c)+((-q)^{a-1}-1)D_{n,h}(a-1,b+2,c)$ are

(8.10)
$$\begin{cases} (-q)^{a-1}\{\kappa_{a-1,t-1}(-q)^{(a+b-h+1)(t-1)}-(-q)^{2a+b-h-1}\kappa_{a-1,t-2}(-q)^{(a+b-h+1)(t-2)}\}X_0^{[t]}, & \text{if } t\neq a,\\ -(-q)^{3a+b-h-2}\kappa_{a-1,a-2}(-q)^{(a+b-h+1)(a-2)}X_0^{[a]} & \text{if } t=a.\end{cases}$$

Similarly, for $1\leq t\leq a$, $X_1^{[t]}$-terms in $-(-q)^{a-1}D_{n,h}(a,b,c+1)$ are

(8.11)
$$\begin{cases} -(-q)^{a-1}\{\kappa_{a-1,t-1}(-q)^{(a+b-h+1)(t-1)}-(-q)^{2a+b-h-1}\kappa_{a-1,t-2}(-q)^{(a+b-h+1)(t-2)}\}X_1^{[t]}, & \text{if } t\neq a,\\ (-q)^{3a+b-h-2}\kappa_{a-1,a-2}(-q)^{(a+b-h+1)(a-2)}X_1^{[a]} & \text{if } t=a.\end{cases}$$

Finally, for $1\leq t\leq a$, $X_2^{[t]}$-terms in $(-q)^{2a+b-1}(1-(-q)^b)D_{n,h}(a,b-1,c+2)+(-q)^{2a+2b-1}(1-(-q)^{a-1})D_{n,h}(a-1,b,c+2)$ are

(8.12)
$$\begin{cases} \{\kappa_{a-1,t-1}(-q)^{(a+b-h)(t-1)+2a+b-1}(1-(-q)^{a+b-1})\\ \quad -\kappa_{a-1,t-2}(-q)^{(a+b-h)(t-2)+4a+2b-h-3}(1-(-q)^b)\}X_2^{[t]} & \text{if } t\neq a,\\ -\kappa_{a-1,a-2}(-q)^{(a+b-h)(a-2)+4a+2b-h-3}(1-(-q)^b)X_2^{[a]} & \text{if } t=a.\end{cases}$$

First, assume that $a+b\geq h+2$. Then, by (8.6), (8.10), (8.11), (8.12), and (8.8), we have that the degree t-terms of $F_{n,h}(a,b,c)$ are 0 for $1\leq t\leq a-1$ (cf. (8.7)).

For $t=a$, we have that the degree a-terms of $F_{n,h}(a,b,c)$ are

$$\prod_{l=h-a+2}^{b-1}(1-(-q)^l)\quad\Big\{-(-q)^{3a+b-h-2}\kappa_{a-1,a-2}(-q)^{(a+b-h+1)(a-2)}(1-(-q)^{b+1})(1-(-q)^b)$$
$$+(-q)^{3a+b-h-2}\kappa_{a-1,a-2}(-q)^{(a+b-h+1)(a-2)}(1-(-q)^b)$$
$$-\kappa_{a-1,a-2}(-q)^{(a+b-h)(a-2)+4a+2b-h-3}(1-(-q)^b)\Big\}=0.$$

Now, the remaining cases are $a+b<h-1$, $a+b=h-1$, h, and $h+1$, and their proofs are similar to that of Lemma 8.14. Therefore, we omit the proofs. □

Lemma 8.17. *Assume that $a=1$. Then, we have*

$$F_{n,h}(a,b,c)=\begin{cases}0 & \text{if } b\geq h+1 \text{ or } b\leq h-3,\\ 1 & \text{if } b=h-2,\\ 0 & \text{if } b=h-1,\\ q^{2h+1} & \text{if } b=h.\end{cases}$$

Proof. This follows from Theorem 7.8 □

Theorem 8.18. *Assume that $x\perp L^\flat$, $\mathrm{val}((x,x))\leq -2$, and $L^\flat\subset L'^\flat\subset (L'^\flat)^\vee\subset L_F^\flat$. Let λ be the fundamental invariants of $L'^\flat$ and $(a,b,c)=(t_{\geq 2}(\lambda),t_1(\lambda),t_0(\lambda))$. Assume further that $(a,b,c)\neq(1,h,n-h-2),(1,h-2,n-h),(0,h+1,n-h-2),(0,h,n-h-1),(0,h-1,n-h)$. Then, we have*

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x)=0.$$

Proof. Recall that by (8.1), we have

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x)=\sum_{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times}D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(L'^\flat+\langle u^\flat+u^\perp\rangle).$$

Since $\mathrm{val}((x,x))\leq -2$, we have that $\mathrm{val}((u^\perp,u^\perp))\geq 2$. Therefore, by Proposition 8.3, $D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)$ depends only on $u^\flat$. Also, we have that $\mathrm{vol}(L'^\flat+\langle u^\flat+u^\perp\rangle)=\mathrm{vol}(L'^\flat)\,\mathrm{vol}(\langle u^\perp\rangle)$.

This implies that

$$\begin{aligned}\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x)&=\sum_{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times}D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(L'^\flat+\langle u^\flat+u^\perp\rangle)\\&=\mathrm{vol}(L'^\flat)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}\sum_{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat}D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle).\end{aligned}$$

Now, it suffices to show that $\displaystyle\sum_{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat}D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)=0$.

We claim that

(8.13)
$$\sum_{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat}D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)$$
$$=\mu(L_{(\lambda\geq 2)-2})\times E_{n,h}(a,b,c)+\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}\times F_{n,h}(a,b,c).$$

Note that if the claim (8.13) holds, the theorem follows from Lemmas 8.14, 8.15, 8.16 and 8.17.

Now we prove the claim. Note that both sides of (8.13) are linear sums of the Cho-Yamauchi constants $D_{n,h}(a+1,b,c)$, $D_{n,h}(a,b+1,c)$, $D_{n,h}(a,b,c+1)$, $D_{n,h}(a+1,b-2,c+2)$, $D_{n,h}(a,b-1,c+2)$, $D_{n,h}(a-1,b+2,c)$, $D_{n,h}(a-1,b,c+2)$. Therefore, it suffices to show that both sides have the same coefficients.

First, consider the coefficients of $D_{n,h}(a+1,b,c)$ on both sides. By Proposition 8.7 (Case 1-1), we have that the coefficient of $D_{n,h}(a+1,b,c)$ on the left-hand side of (8.13) is $\mu(L_{(\lambda\geq 2)-2})$. Also, the coefficients of $D_{n,h}(a+1,b,c)$ on the right hand side of (8.13) is $\mu(L_{\lambda\geq 2-2})$ (note that $D_{n,h}(a+1,b,c)$ appears only in $E_{n,h}(a,b,c)$). Therefore, the coefficients of $D_{n,h}(a+1,b,c)$ on both sides are the same.

For $D_{n,h}(a,b+1,c)$, by Proposition 8.7 (Case 1-2), we have that the coefficient of $D_{n,h}(a,b+1,c)$ on the left hand side of (8.13) is $q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-\mu(L_{(\lambda\geq 2)-2})$. Now, the coefficient of $D_{n,h}(a,b+1,c)$ on the right hand side of (8.13) is

$$\begin{aligned}&\mu(L_{(\lambda\geq 2)-2})\{(-q)^{t_{\geq 2}(\lambda)}-1\}+\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}\\&=q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-\mu(L_{(\lambda\geq 2)-2}).\end{aligned}$$

Therefore, the coefficients of $D_{n,h}(a,b+1,c)$ on both sides are the same.

For $D_{n,h}(a,b,c+1)$, by Proposition 8.7 (Case 1-3+Case 3-2), we have that the coefficient of $D_{n,h}(a,b,c+1)$ on the left hand side of (8.13) is $q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})-\mu(L_{(\lambda\geq 2)-1})$. On the other hand, the coefficient of $D_{n,h}(a,b,c+1)$ on the right hand side of (8.13) is

$$\mu(L_{(\lambda\geq 2)-2})(-q)^{t_{\geq 2}(\lambda)}\{(-q)^{t_{\geq 2}(\lambda)}-1\}-(-q)^{t_{\geq 2}(\lambda)-1}\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}.$$

Therefore, it suffices to show that

$$\mu(L_{(\lambda\geq 2)-1})+(-q)^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 2)-2})=(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})+(-q)^{t_{\geq 2}(\lambda)+2t_{\geq 3}(\lambda)-1}\mu(L_{(\lambda\geq 3)-3}).$$

By Lemma 8.11, we have that

$$\mu(L_{(\lambda\geq 2)-1})-q^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 2)-2})=-(q-1)(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1},$$

$$(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})+(-q)^{t_{\geq 2}(\lambda)+2t_{\geq 3}(\lambda)-1}\mu(L_{(\lambda\geq 3)-3})=-(q-1)(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1}.$$

This shows that the coefficients of $D_{n,h}(a,b,c+1)$ on both sides are the same.

For $D_{n,h}(a+1,b-2,c+2)$, by Proposition 8.7 (Case 2-1+Case 4-1), we have that the coefficient of $D_{n,h}(a+1,b-2,c+2)$ on the left hand side of (8.13) is $q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\times(\mu(L_{\lambda=1})-1)$. On the other hand, the coefficient of $D_{n,h}(a+1,b-2,c+2)$ on the right hand side of (8.13) is

$$-(-q)^{2t_{\geq 2}(\lambda)}(1-(-q)^{t_1(\lambda)})(1-(-q)^{t_1(\lambda)-1})\mu(L_{(\lambda\geq 2)-2}).$$

Now, by Lemma 8.11, we have that

$$\mu(L_{\lambda=1})-1=q^{2t_1(\lambda)-1}-(-q)^{t_1(\lambda)-1}(q-1)-1=-(1-(-q)^{t_1(\lambda)})(1-(-q)^{t_1(\lambda)-1}). \tag{8.14}$$

This shows that the coefficients of $D_{n,h}(a+1,b-2,c+2)$ on both sides are the same.

For $D_{n,h}(a,b-1,c+2)$, by Proposition 8.7 (Case 2-2+Case 4-2), we have that the coefficient of $D_{n,h}(a,b-1,c+2)$ on the left hand side of (8.13) is

$$q^{2t_{\geq 2}(\lambda)}\{\mu(L_{(\lambda\geq 2)-2}\oplus L_{\lambda=1})-\mu(L_{(\lambda\geq 2)-2})\mu(L_{\lambda=1})\}.$$

On the other hand, the coefficient of $D_{n,h}(a, b-1, c+2)$ on the right hand side is

$$-(-q)^{2t_{\geq 2}(\lambda)+t_1(\lambda)-1}(1-(-q)^{t_1(\lambda)})(1-(-q)^{t_{\geq 2}(\lambda)})\mu(L_{(\lambda\geq 2)-2})$$
$$+(-q)^{2t_{\geq 2}(\lambda)+t_1(\lambda)-1}(1-(-q)^{t_1(\lambda)})\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}.$$

Note that by Lemma 8.11, we have that

$$\mu(L_{(\lambda\geq 2)-2}\oplus L_{\lambda=1})=\mu(L_{(\lambda\geq 3)-2}\oplus L_{\lambda=1})=q^{2t_{\geq 3}(\lambda)+2t_1(\lambda)-1}\mu(L_{(\lambda\geq 3)-3})-(q-1)(-q)^{|\lambda|-2t_{\geq 2}(\lambda)-1}.$$

Combining these with (8.14), it suffices to show that

$$-(q-1)(-q)^{|\lambda|-1}=(-q)^{2t_{\geq 2}(\lambda)+t_1(\lambda)}\mu(L_{(\lambda\geq 2)-2})+(-q)^{2t_{\geq 3}(\lambda)+2t_{\geq 2}(\lambda)+t_1(\lambda)-1}\mu(L_{(\lambda\geq 3)-3}).$$

Since $\mu(L_{(\lambda\geq 2)-2})=\mu(L_{(\lambda\geq 3)-2})$, this follows from Lemma 8.11. This shows that the coefficients of $D_{n,h}(a, b-1, c+2)$ on both sides are the same.

For $D_{n,h}(a-1, b+2, c)$, by Proposition 8.7 (Case 3-1), we have that the coefficient of $D_{n,h}(a-1, b+2, c)$ on the left hand side of (8.13) is $\mu(L_{(\lambda\geq 2)-1})-q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})$. On the other hand, the coefficient of $D_{n,h}(a-1, b+2, c)$ on the right hand side of (8.13) is

$$((-q)^{t_{\geq 2}(\lambda)-1}-1)\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}.$$

Therefore, it suffices to show that

$$\mu(L_{(\lambda\geq 2)-1})+(-q)^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 2)-2})=(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})+(-q)^{2t_{\geq 3}(\lambda)+t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 3)-3}).$$

Now, by Lemma 8.11, we have that

$$\mu(L_{(\lambda\geq 2)-1})+(-q)^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 2)-2})=-(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1}(q-1),$$

$$(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})+(-q)^{2t_{\geq 3}(\lambda)+t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 3)-3})=(-q)^{t_{\geq 2}(\lambda)}(-(-q)^{|\lambda|-t_1(\lambda)-2t_{\geq 2}(\lambda)-1})(q-1).$$

This shows that the coefficients of $D_{n,h}(a-1, b+2, c)$ on both sides are the same.

For $D_{n,h}(a-1, b, c+2)$, by Proposition 8.7 (Case 5), we have that the coefficient of $D_{n,h}(a-1, b, c+2)$ on the left hand side of (8.13) is $\mu(L_{\lambda\geq 2}\oplus L_{\lambda=1})-q^{2t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2}\oplus L_{\lambda=1})$. On the other hand, the coefficient of $D_{n,h}(a-1, b, c+2)$ on the right hand side of (8.13) is

$$(-q)^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1}(1-(-q)^{t_{\geq 2}(\lambda)-1})\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}.$$

Note that by Lemma 8.11, we have

$$\mu(L_{\lambda\geq 2}\oplus L_{\lambda=1})=q^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1}\mu(L_{(\lambda\geq 2)-1})-(q-1)(-q)^{|\lambda|-1},$$

$$\mu(L_{(\lambda\geq 2)-2}\oplus L_{\lambda=1})=\mu(L_{(\lambda\geq 3)-2}\oplus L_{\lambda=1})=q^{2t_{\geq 3}(\lambda)+2t_1(\lambda)-1}\mu(L_{(\lambda\geq 3)-3})-(q-1)(-q)^{|\lambda|-2t_{\geq 2}(\lambda)-1}.$$

Therefore, it suffices to show that

$$q^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1}\mu(L_{(\lambda\geq 2)-1})-q^{2t_{\geq 3}(\lambda)+2t_{\geq 2}(\lambda)+2t_1(\lambda)-1}\mu(L_{(\lambda\geq 3)-3})$$
$$=(-q)^{2t_{\geq 2}(\lambda)+2t_1(\lambda)-1}(1-(-q)^{t_{\geq 2}(\lambda)-1})\{q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3})-(-q)^{t_{\geq 2}(\lambda)}\mu(L_{(\lambda\geq 2)-2})\}.$$

Note that this is equivalent to

$$\mu(L_{(\lambda\geq 2)-1})-q^{2t_{\geq 2}(\lambda)-1}\mu(L_{(\lambda\geq 2)-2})=(-q)^{t_{\geq 2}(\lambda)}\{\mu(L_{(\lambda\geq 3)-2})-q^{2t_{\geq 3}(\lambda)-1}\mu(L_{(\lambda\geq 3)-3})\}.$$

By Lemma 8.11, both sides are equal to $-(-q)^{|\lambda|-t_1(\lambda)-t_{\geq 2}(\lambda)-1}(q-1)$. This finishes the proof of (8.13). The theorem follows from Lemma 8.14, Lemma 8.16, Lemma 8.15, and Lemma 8.17. □

Theorem 8.19. *Assume that $x \perp L^\flat$, $\mathrm{val}(\langle x,x\rangle) \le -2$, and $L^\flat \subset L'^\flat \subset (L'^\flat)^\vee \subset L_F^\flat$. Let λ be the fundamental invariants of $L'^\flat$ and $(a,b,c) = (t_{\ge 2}(\lambda), t_1(\lambda), t_0(\lambda))$. Then, we have*

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \begin{cases} q^{h-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (1,h,n-h-2), \\ q^{-h}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (1,h-2,n-h), \\ q^{-(h-1)}\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (0,h-1,n-h), \\ q^{-h}\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (0,h,n-h-1), \\ q^{-(h+1)}(q^{2h+1}+(-q)^h)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (0,h+1,n-h-2). \end{cases}$$

Proof. In the proof of Theorem 8.18, we proved that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \mathrm{vol}(L'^\flat)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} \sum_{u^\flat \in (L'^\flat)^{\vee,\ge 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle).$$

Also, in (8.13), we showed that

$$\sum_{u^\flat \in (L'^\flat)^{\vee,\ge 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle)$$
$$= \mu(L_{(\lambda\ge 2)-2}) \times E_{n,h}(a,b,c) + \{q^{2t_{\ge 3}(\lambda)}\mu(L_{(\lambda\ge 3)-3}) - (-q)^{t_{\ge 2}(\lambda)}\mu(L_{(\lambda\ge 2)-2})\} \times F_{n,h}(a,b,c).$$

First, assume that $(a,b,c) = (1,h,n-h-2)$, $\lambda = (\alpha, \overbrace{1,\dots,1}^{h}, \overbrace{0,\dots,0}^{n-h-2})$. In this case, we know that $E_{n,h}(a,b,c) = 0$. Also, it is easy to see that $\mu(L_\alpha) = q^{2[\frac{\alpha}{2}]}$, and hence

$$q^{2t_{\ge 3}(\lambda)}\mu(L_{(\lambda\ge 3)-3}) - (-q)^{t_{\ge 2}(\lambda)}\mu(L_{(\lambda\ge 2)-2}) = q^{2t_{\ge 3}(\alpha)}\mu(L_{\alpha-3}) + q\mu(L_{\alpha-2}) = q^{\alpha-2}(q+1).$$

Now, by Lemma 8.17, we have that $\sum_{u^\flat \in (L'^\flat)^{\vee,\ge 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = q^{\alpha-2}(q+1)q^{2h+1}$.

Similarly, for $(a,b,c) = (1,h-2,n-h)$, $\lambda = (\alpha, \overbrace{1,\dots,1}^{h-2}, \overbrace{0,\dots,0}^{n-h})$, we have that

$$\sum_{u^\flat \in (L'^\flat)^{\vee,\ge 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = q^{\alpha-2}(q+1).$$

Therefore, we have that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \mathrm{vol}(L'^\flat)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} \sum_{u^\flat \in (L'^\flat)^{\vee,\ge 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle)$$
$$= \begin{cases} q^{h-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (1,h,n-h-2), \\ q^{-h}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } (a,b,c) = (1,h-2,n-h). \end{cases}$$

Now, assume that $a = 0$. Then, we have that

$$\mu(L_{(\lambda\ge 2)-2}) = 1, q^{2t_{\ge 3}(\lambda)}(L_{(\lambda\ge 3)-3}) - (-q)^{t_{\ge 2}(\lambda)}\mu(L_{(\lambda\ge 2)-2}) = 0.$$

Therefore, by Lemma 8.15, we have

$$\sum_{u^\flat \in (L'^\flat)^{\vee,\ge 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = \begin{cases} 1 & \text{if } b = h-1, h, \\ q^{2h+1} + (-q)^h & \text{if } b = h+1. \end{cases}$$

This implies that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \mathrm{vol}(L'^{\flat})\,\mathrm{vol}(\langle x\rangle^{\vee})(1-q^{-2})^{-1}\sum_{u^{\flat}\in(L'^{\flat})^{\vee,\geq 0}/L'^{\flat}} D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)$$
$$= \begin{cases} q^{-(h-1)}\,\mathrm{vol}(\langle x\rangle^{\vee})(1-q^{-2})^{-1} & \text{if } (a,b,c)=(0,h-1,n-h),\\ q^{-h}\,\mathrm{vol}(\langle x\rangle^{\vee})(1-q^{-2})^{-1} & \text{if } (a,b,c)=(0,h,n-h-1),\\ q^{-(h+1)}(q^{2h+1}+(-q)^{h})\,\mathrm{vol}(\langle x\rangle^{\vee})(1-q^{-2})^{-1} & \text{if } (a,b,c)=(0,h+1,n-h-2).\end{cases}$$

This finishes the proof of the theorem. □

Theorem 8.20. *Assume that $x\perp L^{\flat}$, $\mathrm{val}((x,x))=-1$, and $L^{\flat}\subset L'^{\flat}\subset (L'^{\flat})^{\vee}\subset L_F^{\flat}$. Let λ be the fundamental invariants of $L'^{\flat}$ and $(a,b,c)=(t_{\geq 2}(\lambda),t_1(\lambda),t_0(\lambda))$. Assume further that $(a,b,c)\neq(1,h,n-h-2),(1,h-2,n-h),(0,h+1,n-h-2),(0,h,n-h-1),(0,h-1,n-h)$. Then, we have*

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = -\frac{1}{q^h}D_{n-1,h-1}(a,b,c).$$

Proof. Recall that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \sum_{(u^{\flat},u^{\perp})\in(L'^{\flat})^{\vee,\geq 0}/L'^{\flat}\times(\langle x\rangle^{\vee}\setminus\{0\})/O_F^{\times}} D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)\,\mathrm{vol}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle).$$

Since $\mathrm{val}((x,x))=-1$, we have that $\mathrm{val}((u^{\perp},u^{\perp}))\geq 1$. In the proof of Theorem 8.18, we have

$$\sum_{(u^{\flat},u^{\perp})\in(L'^{\flat})^{\vee,\geq 0}/L'^{\flat}\times(\langle x\rangle^{\vee}\setminus\{0\})/O_F^{\times},\mathrm{val}((u^{\perp},u^{\perp}))\geq 2} D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)\,\mathrm{vol}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)=0.$$

Therefore, we have that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \sum_{\substack{(u^{\flat},u^{\perp})\in(L'^{\flat})^{\vee,\geq 0}/L'^{\flat}\times(\langle x\rangle^{\vee}\setminus\{0\})/O_F^{\times},\\ \mathrm{val}((u^{\perp},u^{\perp})=1}} D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)\,\mathrm{vol}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle).$$

Then by Proposition 8.4, $D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)$ depends only on $u^{\flat}$. Also, we have that $\mathrm{vol}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)=\mathrm{vol}(L'^{\flat})\,\mathrm{vol}(\langle u^{\perp}\rangle)$. This implies that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \sum_{\substack{(u^{\flat},u^{\perp})\in(L'^{\flat})^{\vee,\geq 0}/L'^{\flat}\times(\langle x\rangle^{\vee}\setminus\{0\})/O_F^{\times},\\ \mathrm{val}((u^{\perp},u^{\perp}))=1}} D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)\,\mathrm{vol}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle)$$
$$= \mathrm{vol}(L'^{\flat})(q)^{-1}\sum_{u^{\flat}\in(L'^{\flat})^{\vee,\geq 0}/L'^{\flat}} D_{n,h}(L'^{\flat}+\langle u^{\flat}+u^{\perp}\rangle).$$

Note that Proposition 8.3 and Proposition 8.4 are different only when (Case 1-1) and (Case 1-2). Also, it is easy to see that

$$\begin{aligned}
&\text{(Case 1-2-1)}\\
&|\{u^\flat \in (\pi^2(L_2'^\flat)^\vee \oplus \pi(L_1'^\flat)^\vee)^{\geq 1} - (\pi^2(L_2'^\flat)^\vee \oplus \pi(L_1'^\flat)^\vee)^{\geq 2} \mid \mathrm{val}((u^\flat + u^\perp, u^\flat + u^\perp)) = 1\}|\\
&= \frac{q-2}{q-1}(\text{Case 1-2}) = \frac{q-2}{q-1}(q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3}) - \mu(L_{(\lambda\geq 2)-2})),\\
&\text{(Case 1-2-2)}\\
&|\{u^\flat \in (\pi^2(L_2'^\flat)^\vee \oplus \pi(L_1'^\flat)^\vee)^{\geq 1} - (\pi^2(L_2'^\flat)^\vee \oplus \pi(L_1'^\flat)^\vee)^{\geq 2} \mid \mathrm{val}((u^\flat + u^\perp, u^\flat + u^\perp)) \geq 2\}|\\
&= \frac{1}{q-1}(\text{Case 1-2}) = \frac{1}{q-1}(q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3}) - \mu(L_{(\lambda\geq 2)-2})).
\end{aligned}$$

Now, note that if $\mathrm{val}((u^\perp, u^\perp)) \geq 2$, then $\sum_{u^\flat \in (L'^\flat)^{\vee,\geq 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle) = 0$. Therefore, for $\mathrm{val}((u^\perp, u^\perp)) = 1$, we have

$$\begin{aligned}
&\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \frac{1}{q}\,\mathrm{vol}(L'^\flat) \sum_{u^\flat \in (L'^\flat)^{\vee,\geq 0}/L'^\flat} D_{n,h}(L'^\flat + \langle u^\flat + u^\perp\rangle)\\
&= \frac{1}{q}\,\mathrm{vol}(L'^\flat)\{(D_{n,h}(a,b+1,c) - D_{n,h}(a+1,b,c))\mu(L_{(\lambda\geq 2)-2})\\
&\qquad +(D_{n,h}(a+1,b,c) - D_{n,h}(a,b+1,c))\frac{1}{q-1}(q^{2t_{\geq 3}(\lambda)}\mu(L_{(\lambda\geq 3)-3}) - \mu(L_{(\lambda\geq 2)-2}))\}\\
&= \frac{1}{q}\,\mathrm{vol}(L'^\flat)(D_{n,h}(a+1,b,c) - D_{n,h}(a,b+1,c))\{\frac{q^{2t_{\geq 3}(\lambda)}}{q-1}\mu(L_{(\lambda\geq 3)-3}) - \frac{q}{q-1}\mu(L_{(\lambda\geq 2)-2})\}.
\end{aligned}$$

By Lemma 8.11, we have $\frac{q^{2t_{\geq 3}(\lambda)}}{q-1}\mu(L_{(\lambda\geq 3)-3}) - \frac{q}{q-1}\mu(L_{(\lambda\geq 2)-2}) = -(-q)^{|\lambda|-b-2a}$.
Also, by Theorem 7.4, we have $D_{n,h}(a+1,b,c) - D_{n,h}(a,b+1,c) = -(-q)^{2n-h-1-b-2c}D_{n-1,h-1}(a,b,c)$.
Finally, we have that $\mathrm{vol}(L'^\flat) = q^{-|\lambda|}$. Therefore, we have

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = \frac{(-q)^{|\lambda|+2n-2a-2b-2c-h-1}}{q^{|\lambda|+1}}D_{n-1,h-1}(a,b,c).$$

Now, note that $a+b+c = n-1$ and $\mathrm{val}(L'^\flat \oplus \langle x\rangle) = |\lambda| - 1 \equiv h+1 \pmod 2$. This implies that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) = -\frac{1}{q^h}D_{n-1,h-1}(a,b,c).$$

This finishes the proof of the theorem. □

Theorem 8.21. *Assume that $x \perp L^\flat$, $\mathrm{val}((x,x)) = -1$, and $L^\flat \subset L'^\flat \subset (L'^\flat)^\vee \subset L_F^\flat$. Let λ be the fundamental invariants of $L'^\flat$ and $(a,b,c) = (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda))$. Then, we have*

$$\begin{aligned}
&\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x)\\
&= \begin{cases}
q^{h-2}(q+1)(1-q^{-2})^{-1} - q^{-h}D_{n-1,h-1}(a,b,c) & \text{if } (a,b,c) = (1,h,n-h-2),\\
q^{-h-1}(q+1)(1-q^{-2})^{-1} - q^{-h}D_{n-1,h-1}(a,b,c) & \text{if } (a,b,c) = (1,h-2,n-h),\\
q^{-h-1}(1-q^{-2})^{-1} - q^{-h}D_{n-1,h-1}(a,b,c) & \text{if } (a,b,c) = (0,h,n-h-1).
\end{cases}
\end{aligned}$$

Note that two cases $(a,b,c) = (0,h-1,n-h), (0,h+1,n-h-2)$ are not possible since $\mathrm{val}((x,x)) = -1$ and hence $\mathrm{val}(L^\flat) \equiv h \pmod 2$.

Proof. Recall that

$$\begin{aligned}\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) &= \sum_{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(L'^\flat+\langle u^\flat+u^\perp\rangle)\\ &= \mathrm{vol}(L'^\flat)\sum_{\substack{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times\\ \mathrm{val}((u^\perp,u^\perp)=1}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(\langle u^\perp\rangle)\\ &+ \mathrm{vol}(L'^\flat)\sum_{\substack{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times\\ \mathrm{val}((u^\perp,u^\perp)\geq 3}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(\langle u^\perp\rangle).\end{aligned}\tag{8.15}$$

In the proof of the Theorem 8.19, we showed that

$$\begin{aligned}&\mathrm{vol}(L'^\flat)\sum_{\substack{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times\\ \mathrm{val}((u^\perp,u^\perp)\geq 3}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(\langle u^\perp\rangle)\\ &= \mathrm{vol}(L'^\flat)\frac{1}{q^3}(1-q^{-2})^{-1}\sum_{\substack{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\\ \mathrm{val}((u^\perp,u^\perp)\geq 3}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle).\end{aligned}$$

Also, we already computed the sum $\sum_{\substack{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\\ \mathrm{val}((u^\perp,u^\perp)\geq 3}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)$ in the proof of the Theorem 8.19 (more precisely, we computed this for $\mathrm{val}((u^\perp,u^\perp))\geq 2$). Therefore, we only need to compute

$$\mathrm{vol}(L'^\flat)\sum_{\substack{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times\\ \mathrm{val}((u^\perp,u^\perp)=1}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(\langle u^\perp\rangle).$$

Note that Proposition 8.3 and Proposition 8.4 are different only when (Case 1-1) and (Case 1-2). Therefore, as in the proof of Theorem 8.20, we have that

$$\begin{aligned}&\sum_{\substack{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\\ \mathrm{val}((u^\perp,u^\perp)=1}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle) - \sum_{\substack{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\\ \mathrm{val}((u^\perp,u^\perp)\geq 2}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\\ &= (D_{n,h}(a+1,b,c)-D_{n,h}(a,b+1,c))\{\frac{q^{2t_{\geq 3}(\lambda)}}{q-1}\mu(L_{(\lambda\geq 3)-3})-\frac{q}{q-1}\mu(L_{(\lambda\geq 2)-2})\}\\ &= (-q)^{|\lambda|-h+1}D_{n-1,h-1}(a,b,c).\end{aligned}$$

Since $\mathrm{vol}(L'^\flat)=q^{-|\lambda|}$ and $\mathrm{val}(L'^\flat\oplus\langle x\rangle)=|\lambda|-1\equiv h+1 \pmod 2$, we have that

$$\begin{aligned}&\mathrm{vol}(L'^\flat)\sum_{\substack{(u^\flat,u^\perp)\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\times(\langle x\rangle^\vee\setminus\{0\})/O_F^\times\\ \mathrm{val}(\langle u^\perp,u^\perp\rangle=1}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)\,\mathrm{vol}(\langle u^\perp\rangle)\\ &= -\frac{1}{q^h}D_{n-1,h-1}(a,b,c)+\frac{\mathrm{vol}(L'^\flat)}{q}\sum_{\substack{u^\flat\in(L'^\flat)^{\vee,\geq 0}/L'^\flat\\ \mathrm{val}((u^\perp,u^\perp)\geq 2}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle).\end{aligned}$$

Now, the theorem follows from (8.15) and the fact that (see the proof of Theorem 8.19)

$$\sum_{\substack{u^\flat\in(L'^\flat)^\vee,\geq 0/L'^\flat\\ \mathrm{val}((u^\perp,u^\perp)\geq 2}} D_{n,h}(L'^\flat+\langle u^\flat+u^\perp\rangle)=\begin{cases} q^{h-1}(q+1)q^{|\lambda|} & \text{if } (a,b,c)=(1,h,n-h-2),\\ q^{-h}(q+1)q^{|\lambda|} & \text{if } (a,b,c)=(1,h-2,n-h-1),\\ 1 & \text{if } (a,b,c)=(0,h,n-h).\end{cases}$$

□

We will use the following lemma in the next section when we count the number of horizontal components. Recall that for $\lambda\in\mathcal{R}_n^{0+}$, we define A_λ by $\mathrm{Diag}(\pi^{\lambda_1},\dots,\pi^{\lambda_n})$.

Lemma 8.22. *Assume that $\lambda\geq 2$. Consider $(\lambda,0^{n-1}),(\lambda,2^{n-1})\in\mathcal{R}_n^{0+}$ and $(0^{n-1}),(2^{n-1})\in\mathcal{R}_{n-1}^{0+}$. Then, we have*

$$\frac{\mathrm{Den}(A_{(\lambda,0^{n-1})},A_{(\lambda,2^{n-1})})/\mathrm{Den}(A_{(\lambda,0^{n-1})},A_{(\lambda,0^{n-1})})}{\mathrm{Den}(A_{(0^{n-1})},A_{(2^{n-1})})/\mathrm{Den}(A_{(0^{n-1})},A_{(0^{n-1})})}=\begin{cases} q^{2n-2} & \text{if } \lambda\geq 3,\\ q^{2n-2}\dfrac{1-(-q)^{-n}}{1-(-q)^{-1}} & \text{if } \lambda=2.\end{cases}$$

Proof. We use [Cho22a, Theorem 2.5] to prove this. To simplify notation, we use the following convention: For $1\leq k\leq n$, $B_1\in X_n(O_F)$, and $B_2\in X_{n-1}(O_F)$, we define

$$\begin{aligned} &R_0^k=\mathrm{Den}(A_{(\lambda,1^{k-1},0^{n-k})},B_1),\quad R_1^k=\mathrm{Den}(A_{(\lambda,2^{k-1},1^{n-k})},B_1),\quad R_2^k=\mathrm{Den}(A_{(\lambda,3^{k-1},2^{n-k})},B_1),\\ &R_3^k=\mathrm{Den}(A_{(1^{k-1},0^{n-k})},B_2),\quad R_4^k=\mathrm{Den}(A_{(2^{k-1},1^{n-k})},B_2),\quad R_5^k=\mathrm{Den}(A_{(3^{k-1},2^{n-k})},B_2),\end{aligned}$$

and for a polynomial $f(X)=\sum_{i=1}^n a_iX^i$, we denote by $f(R_i)$ the sum $f(R_i):=\sum_{i=1}^n a_iR_i^k$. For example, if $f(X)=X+X^2$, then $f(R_1)=\mathrm{Den}(A_{(\lambda,0^{n-1})},B_1)+\mathrm{Den}(A_{(\lambda,1^1,0^{n-2})},B_1)$ (not $\mathrm{Den}(A_{(\lambda,0^{n-1})},B_1)+\{\mathrm{Den}(A_{(\lambda,0^{n-1})},B_1)\}^2$).

Also, we define the following polynomials

$$\begin{aligned} &f_{1,k}(X)=X^k\textstyle\prod_{l=k+1}^n(1-(-q)^{-l}X),\quad f_{2,k}(X)=X\prod_{l=k+1}^n(1-(-q)^{-l}X),\\ &f_{3,k}(X)=X^k\textstyle\prod_{l=k}^{n-1}(1-(-q)^{-l}X),\quad f_{4,k}(X)=X\prod_{l=k}^{n-1}(1-(-q)^{-l}X).\end{aligned}$$

Finally, for $0\leq j\leq i$, we define the following constants $k_{i,j}$: $\prod_{l=1}^i(1+(-q)^{-l}X)=\sum_{j=0}^i k_{i,j}X^j$.

First, we claim that

$$f_{2,i+1}(X)=\sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}}f_{1,j+1}(X),0\leq i\leq n-1,\text{ and} \tag{8.16}$$

$$f_{4,i+1}(X)=\sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j-1)}{2}}f_{3,j+1}(X),0\leq i\leq n-1. \tag{8.17}$$

Let us prove (8.16) by induction on i. For $i=0$, the claim holds since $f_{2,1}(X)=f_{1,1}(X)$. Now, assume that the claim holds for i, i.e.,

$$X\prod_{l=i+2}^n(1-(-q)^{-l}X)=\sum_{j=0}^i k_{i,j}(-q)^{-\frac{j(j+1)}{2}}X^{j+1}\prod_{l=j+2}^n(1-(-q)^{-l}X). \tag{8.18}$$

Note that $\sum_{j=0}^{i+1} k_{i+1,j}X^j = \prod_{l=1}^{i+1}(1+(-q)^{-l}X) = (1+(-q)^{-i-1}X)(\sum_{j=0}^{i} k_{i,j}X^j)$, and hence

$$\begin{aligned}\sum_{j=0}^{i+1} k_{i+1,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X) &= \sum_{j=0}^{i} k_{i,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X) \\ &\quad + \sum_{j=0}^{i}(-q)^{-i-1}k_{i,j}(-q)^{-\frac{(j+1)(j+2)}{2}} f_{1,j+2}(X).\end{aligned} \tag{8.19}$$

Now, consider the equation (8.18) with $X \Rightarrow (-q)^{-1}X$, $n \Rightarrow n-1$. Then, we have

$$\begin{aligned}&(-q)^{-1}X\textstyle\prod_{l=i+3}^{n}(1-(-q)^{-l}X) = \sum_{j=0}^{i} k_{i,j}(-q)^{-\frac{(j+1)(j+2)}{2}} X^{j+1}\prod_{l=j+3}^{n}(1-(-q)^{-l}X) \\ \iff\ &\textstyle\sum_{j=0}^{i}(-q)^{-i-1}k_{i,j}(-q)^{-\frac{(j+1)(j+2)}{2}} f_{1,j+2}(X) = (-q)^{-i-2}X^2\prod_{l=i+3}^{n}(1-(-q)^{-l}X).\end{aligned}$$

Combining this with (8.19), we have $\sum_{j=0}^{i+1} k_{i+1,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(X) = X\prod_{l=i+3}^{n}(1-(-q)^{-l}X)$. This finishes the proof of (8.16). The equation (8.17) can be proved in a similar way.

Now, note that [Cho22a, Theorem 2.5] implies that if $B_1 = A_{\eta_1}$, $B_2 = A_{\eta_2}$ such that $\eta_1 \geq (\overbrace{2,\dots,2}^{n})$, $\eta_2 \geq (\overbrace{2,\dots,2}^{n-1})$, then we have

$$\begin{aligned} f_{1,j}(R_0) &= \frac{(-1)^{n-j}}{(-q)^{n(n-j)}} f_{2,j}(R_1), \qquad f_{1,j}(R_1) = \frac{(-1)^{n-j}}{(-q)^{n(n-j)}} f_{2,j}(R_2), \\ f_{3,j}(R_3) &= \frac{(-1)^{n-j}}{(-q)^{(n-1)(n-j)}} f_{4,j}(R_4), \quad f_{3,j}(R_4) = \frac{(-1)^{n-j}}{(-q)^{(n-1)(n-j)}} f_{4,j}(R_5). \end{aligned} \tag{8.20}$$

Furthermore, if $B_1 = A_{(\lambda,2^{n-1})}$ and $B_2 = A_{(2^{n-1})}$, then $R_2^k = R_5^k = 0$ for all $k > 1$. Therefore,

$$f_{2,j}(R_2) = R_2 = \mathrm{Den}(A_{(\lambda,2^{n-1})}, A_{(\lambda,2^{n-1})}), \quad f_{4,j}(R_5) = R_5 = \mathrm{Den}(A_{(2^{n-1})}, A_{(2^{n-1})}). \tag{8.21}$$

Since $\mathrm{Den}(A_{(\lambda,1,0^{n-2})}, A_{(\lambda,2^{n-1})}) = 0$, we have

(8.22)

$$\begin{aligned} &\mathrm{Den}(A_{(\lambda,0^{n-1})}, A_{(\lambda,2^{n-1})}) = R_0(1-(-q)^{-n}R_0) = f_{2,n-1}(R_0) \\ &= \textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} f_{1,i+1}(R_0) && \text{(by (8.16))} \\ &= \textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{n(n-i-1)}} f_{2,i+1}(R_1) && \text{(by (8.20))} \\ &= \textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{n(n-i-1)}} \sum_{j=0}^{i} k_{i,j}(-q)^{-\frac{j(j+1)}{2}} f_{1,j+1}(R_1) && \text{(by (8.16))} \\ &= \{\textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i+1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{n(n-i-1)}} \sum_{j=0}^{i} k_{i,j}(-q)^{-\frac{j(j+1)}{2}} \frac{(-1)^{n-j-1}}{(-q)^{n(n-j-1)}}\}R_2 && \text{(by (8.20), (8.21)).} \end{aligned}$$

Similarly, note that $\mathrm{Den}(A_{(1,0^{n-2})}, A_{(2^{n-1})}) = 0$, and hence

(8.23)

$$\begin{aligned} &\mathrm{Den}(A_{(0^{n-1})}, A_{(2^{n-1})}) = R_3(1-(-q)^{-(n-1)}R_3) = f_{4,n-1}(R_3) \\ &= \textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} f_{3,i+1}(R_3) && \text{(by (8.17))} \\ &= \textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{(n-1)(n-i-1)}} f_{4,i+1}(R_4) && \text{(by (8.20))} \\ &= \textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{(n-1)(n-i-1)}} \sum_{j=0}^{i} k_{i,j}(-q)^{-\frac{j(j-1)}{2}} f_{3,j+1}(R_4) && \text{(by (8.17))} \\ &= \{\textstyle\sum_{i=0}^{n-2} k_{n-2,i}(-q)^{-\frac{i(i-1)}{2}} \frac{(-1)^{n-i-1}}{(-q)^{(n-1)(n-i-1)}} \sum_{j=0}^{i} k_{i,j}(-q)^{-\frac{j(j-1)}{2}} \frac{(-1)^{n-j-1}}{(-q)^{(n-1)(n-j-1)}}\}R_5 && \text{(by (8.20), (8.21)).} \end{aligned}$$

Comparing (8.22) and (8.23), we have

$$(-q)^{2n-2}\frac{\mathrm{Den}(A_{(\lambda,0^{n-1})},A_{(\lambda,2^{n-1})})}{\mathrm{Den}(A_{(\lambda,2^{n-1})},A_{(\lambda,2^{n-1})})}=\frac{\mathrm{Den}(A_{(0^{n-1})},A_{(2^{n-1})})}{\mathrm{Den}(A_{(2^{n-1})},A_{(2^{n-1})})}. \tag{8.24}$$

Now, by [Cho23, Lemma 4.6], we have

$$\begin{aligned}&\mathrm{Den}(A_{(2^{n-1})},A_{(2^{n-1})})=q^{2(n-1)^2}\mathrm{Den}(A_{(0^{n-1})},A_{(0^{n-1})})=q^{2(n-1)^2}\textstyle\prod_{l=1}^{n-1}(1-(-q)^{-l}),\\&\mathrm{Den}(A_{\lambda,(0^{n-1})},A_{(\lambda,0^{n-1})})=q^{\lambda}(1-(-q)^{-1})\textstyle\prod_{l=1}^{n-1}(1-(-q)^{-l}),\ \text{if } \lambda\geq 1,\end{aligned}$$

$$\mathrm{Den}(A_{(\lambda,2^{n-1})},A_{(\lambda,2^{n-1})})=\begin{cases}q^{2(n^2-1)+\lambda}(1-(-q)^{-1})\prod_{l=1}^{n-1}(1-(-q)^{-l}) & \text{if } \lambda\geq 3,\\ q^{2n^2}\prod_{l=1}^{n}(1-(-q)^{-l}) & \text{if } \lambda=2.\end{cases}$$

Combining these with (8.24), we have

$$\begin{aligned}&\frac{\mathrm{Den}(A_{(\lambda,0^{n-1})},A_{(\lambda,2^{n-1})})/\mathrm{Den}(A_{(\lambda,0^{n-1})},A_{(\lambda,0^{n-1})})}{\mathrm{Den}(A_{(0^{n-1})},A_{(2^{n-1})})/\mathrm{Den}(A_{(0^{n-1})},A_{(0^{n-1})})}\\&=(-q)^{-2n+2}\frac{\mathrm{Den}(A_{(\lambda,2^{n-1})},A_{(\lambda,2^{n-1})})}{\mathrm{Den}(A_{(\lambda,0^{n-1})},A_{(\lambda,0^{n-1})})}\frac{\mathrm{Den}(A_{(0^{n-1})},A_{(0^{n-1})})}{\mathrm{Den}(A_{(2^{n-1})},A_{(2^{n-1})})}=\begin{cases}q^{2n-2} & \text{if } \lambda\geq 3,\\ q^{2n-2}\dfrac{1-(-q)^{-n}}{1-(-q)^{-1}} & \text{if } \lambda=2.\end{cases}\end{aligned}$$

This finishes the proof of the lemma. □

9. Tate conjectures and the proof of the main theorem

9.1. **The proof of the main theorem.** Consider the Rapoport–Zink space $\mathcal{N}_n^{[h]}$ and the space of special homomorphisms $\mathbb{V}$. Assume that $L^\flat\subset\mathbb{V}$ is an O_F-lattice of rank $n-1$. Recall that for $L'^\flat$ such that $L^\flat\subset L'^\flat\subset(L'^\flat)^\vee\subset L_F^\flat$, we define the primitive part $\mathcal{Z}(L'^\flat)^\circ$ of $\mathcal{Z}(L'^\flat)$ inductively by setting

$$\mathcal{Z}(L'^\flat)^\circ:=\mathcal{Z}(L'^\flat)-\sum_{\substack{L'^\flat\subset L''^\flat\\ L''^\flat\subset(L''^\flat)^\vee\subset L_F^\flat}}\mathcal{Z}(L''^\flat)^\circ.$$

For example, for a lattice $L'^\flat$ with fundamental invariants $(0,0,\dots,0,1)$, there is no integral lattice $L''^\flat$ such that $L'^\flat\subset L''^\flat$ and hence $\mathcal{Z}(L'^\flat)^\circ=\mathcal{Z}(L'^\flat)$. For a lattice $L'^\flat$ with fundamental invariants $(0,0,\dots,0,3)$, there is one integral lattice $L''^\flat$ satisfying the above conditions, and its fundamental invariants are $(0,0,\dots,0,1)$. Therefore, we have $\mathcal{Z}(L'^\flat)^\circ=\mathcal{Z}(L'^\flat)-\mathcal{Z}(L''^\flat)^\circ=\mathcal{Z}(L'^\flat)-\mathcal{Z}(L''^\flat)$.

Now, let us define the derived special cycles ${}^{\mathbb{L}}\mathcal{Z}(L)$.

Definition 9.1. For a lattice $L\subset\mathbb{V}$, choose its basis $x_1,\dots,x_r$. We define the derived special cycle ${}^{\mathbb{L}}\mathcal{Z}(L)$ as the image of $O_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}O_{\mathcal{Z}(x_r)}$ in the r-th graded piece of the Grothendieck group $\mathrm{Gr}^rK_0^{\mathcal{Z}(L)}(\mathcal{N})$. This does not depend on the choice of the basis by Proposition 2.11.

By an induction on $\mathrm{val}(L)$, Conjecture 5.7 is equivalent to the following statement: for $x\in\mathbb{V}\backslash L_F^\flat$,

$$\chi(\mathcal{N}_n^{[h]},{}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ\otimes^{\mathbb{L}}O_{\mathcal{Z}(x)})=\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ}}(x):=\sum_{L'^\flat\subset L'\subset L'^\vee,L'\cap L_F^\flat=L'^\flat}D_{n,h}(L')1_{L'}(x), \tag{9.1}$$

where $L'\subset\mathbb{V}$ are O_F-lattices of rank n. For example, for a lattice $L'^\flat$ with fundamental invariants $(0,\dots,0,1)$, it is obvious that Conjecture 5.7 is equivalent to (9.1). For a lattice $L'^\flat$ with

fundamental invariants $(0,\dots,0,3)$, there is one integral lattice $L''^\flat$ such that $L'^\flat \subsetneq L''^\flat$, and the fundamental invariants of $L''^\flat$ is $(0,\dots,0,1)$. Therefore, Conjecture 5.7 is equivalent to

$$\begin{aligned}\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}) &= \sum_{L'^\flat \subset L' \subset L'^\vee} D_{n,h}(L')1_{L'}(x) - \sum_{L'^\flat \subset L' \subset L'^\vee, L' \cap L_F^\flat = L''^\flat} D_{n,h}(L')1_{L'}(x)\\ &= \sum_{L'^\flat \subset L' \subset L'^\vee, L' \cap L_F^\flat = L'^\flat} D_{n,h}(L')1_{L'}(x),\end{aligned}$$

which is equivalent to (9.1).

Now, note that there is a decomposition of the derived special cycle ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)$ into a sum of horizontal and vertical parts (see Section 2.3): ${}^{\mathbb{L}}\mathcal{Z}(L'^\flat) = {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)_{\mathscr{H}} + {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)_{\mathscr{V}}$.

We denote by ${}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}}$ (resp. ${}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{V}}$) the primitive part of ${}^{\mathbb{L}}\mathcal{Z}(L'^\flat)_{\mathscr{H}}$ (resp. ${}^{\mathbb{L}}\mathcal{Z}(L'^\flat)_{\mathscr{V}}$).

We define

$$\begin{aligned}&\mathrm{Int}_{L'^\flat,\mathscr{H}}(x) \coloneqq \chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)_{\mathscr{H}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}), \quad \mathrm{Int}_{L'^\flat,\mathscr{V}}(x) \coloneqq \chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}),\\ &\mathrm{Int}_{L'^{\flat\circ},\mathscr{H}}(x) \coloneqq \chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}), \quad \mathrm{Int}_{L'^{\flat\circ},\mathscr{V}}(x) \coloneqq \chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)}).\end{aligned}$$

Then, Conjecture 5.6 is equivalent to

$$\mathrm{Int}_{L'^\flat,\mathscr{H}}(x) + \mathrm{Int}_{L'^\flat,\mathscr{V}}(x) = \partial\mathrm{Den}^{n,h}_{L'^\flat}(x), \text{ or} \tag{9.2}$$

$$\mathrm{Int}_{L'^{\flat\circ},\mathscr{H}}(x) + \mathrm{Int}_{L'^{\flat\circ},\mathscr{V}}(x) = \partial\mathrm{Den}^{n,h}_{L'^{\flat\circ}}(x). \tag{9.3}$$

Now, we define $\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x)$ by $\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) \coloneqq \partial\mathrm{Den}^{n,h}_{L'^{\flat\circ}}(x) - \mathrm{Int}_{L'^{\flat\circ},\mathscr{H}}(x)$.

The reason we define the vertical part in this way is that it cannot be separated solely by the invariants of $L'^\flat$. For example, even for a certain lattice $L'^\flat \in H(\mathbb{V})$, ${}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ$ consists of both vertical and horizontal parts. Therefore, the analytic part should include both parts as well.

Let $\lambda \in \mathcal{R}^{0+}_{n-1}$ be the fundamental invariants of $L'^\flat$. Then, by Lemma 3.6, we know that if $L'^\flat \notin H(\mathbb{V})$ (i.e., $(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)) \neq (1,h,n-h-2),(1,h-2,n-h),(0,h-1,n-h),(0,h,n-h-1),(0,h+1,n-h-2)$), then we have $\mathrm{Int}_{L'^{\flat\circ},\mathscr{H}}(x) = 0$, and hence we have that $\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) = \partial\mathrm{Den}^{n,h}_{L'^{\flat\circ}}(x)$,in this case.

When $L'^\flat \in H(\mathbb{V})$, then the horizontal part of $\mathcal{Z}(L'^\flat)$ is not empty, and hence we need to be careful. Note that by Theorem 3.3, the horizontal part of $\mathcal{Z}(L'^\flat)^\circ$ comes from $\mathcal{N}_2^{[0]}$ or $\mathcal{N}_2^{[2]}$. By [KR11, Theorem 1.1] and reductions in Proposition 2.6, we have a precise formula for $\mathrm{Int}_{L'^{\flat\circ},\mathscr{H}}(x)$.

Now, we will prove the following theorem.

Theorem 9.2. *(cf. [LZ22a, Theorem 7.4.1]) Assume that $x \perp L'^\flat$ and $(t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)) = (a,b,c)$ where $\lambda \in \mathcal{R}^{0+}_{n-1}$ is the fundamental invariants of $L'^\flat$. Then, we have the followings.*

(1) If $\mathrm{val}((x,x)) \leq -2$, we have $\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) = 0$.

(2) If $\mathrm{val}((x,x)) = -1$, we have

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) = \begin{cases} -\dfrac{1}{q^h} D_{n-1,h-1}(a,b,c) & \text{if } (a,b,c) \neq (1,h-2,n-h),\\ -\dfrac{1}{q^h} D_{n-1,h-1}(a,b,c) + \dfrac{1}{q^h} & \text{if } (a,b,c) = (1,h-2,n-h).\end{cases}$$

Proof. First, if $(a,b,c) := (t_{\geq 2}(\lambda), t_1(\lambda), t_0(\lambda)) \neq (1,h,n-h-2), (1,h-2,n-h), (0,h-1,n-h), (0,h,n-h-1), (0,h+1,n-h-2)$, then the assertion (1) follows from Theorem 8.18, and the assertion (2) follows from Theorem 8.20.

Now, assume that $(a,b,c) = (1,h,n-h-2), (1,h-2,n-h), (0,h-1,n-h), (0,h,n-h-1), (0,h+1,n-h-2)$. Since we already know the Fourier transform of $\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ}}(x)$ by Theorem 8.19 and Theorem 8.21, we only need to know the Fourier transform of $\mathrm{Int}_{L'^{\flat\circ},\mathscr{H}}(x)$.

When $(a,b,c) = (1,h,n-h-2)$, let us write $L'^{\flat} = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are π^λ $(\lambda \geq 2)$, πI_h, I_{n-h-2}, respectively. Then, by Proposition 2.6, $\mathcal{Z}(L'^{\flat})^\circ_{\mathscr{H}}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ in $N_{h+2}^{[h]}$. Therefore, by Theorem 3.3, we have

$$\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}} = \sum_{\substack{L_2 \oplus L_1 \subset L_2 \oplus N \subset \pi^{-1}(L_2 \oplus L_1) \\ N \simeq (\pi^{-1})^h}} \mathcal{Z}(L_2)^\circ \cdot \mathcal{Y}(N)^\circ. \tag{9.4}$$

By Proposition 2.6 again, $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ can be reduced to $\mathcal{Z}(L_2)^\circ_{\mathscr{H}} = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[0]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[0]}$, and hence $\mathrm{Int}_{L_2^\circ,\mathscr{H}}(x) = \mathrm{Int}_{L_2^\circ}(x) = \partial\mathrm{Den}^{2,0}_{L_2^\circ}(x)$.

By Theorem 8.19 and Theorem 8.21, we have that

$$\widehat{\mathrm{Int}}_{L_2^\circ}(x) = \widehat{\partial\mathrm{Den}}^{2,0}_{L_2^\circ}(x) = q^{-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(N_2)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^h}\widehat{\mathrm{Int}}_{L_2^\circ}(x) = q^{-h-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by (1.6) and Lemma 8.22, the number of lattices $L_2 \oplus N$ in the sum in (9.4) is

$$\begin{aligned}&\frac{\mathrm{Den}(A_{(\lambda,(-1)^h)}, A_{(\lambda,1^h)})/\mathrm{Den}(A_{(\lambda,(-1)^h)}, A_{(\lambda,(-1)^h)})}{\mathrm{Den}(A_{((-1)^h)}, A_{(1^h)})/\mathrm{Den}(A_{((-1)^h)}, A_{((-1)^h)})} \\ &= \frac{\mathrm{Den}(A_{((\lambda+1),0^h)}, A_{((\lambda+1),2^h)})/\mathrm{Den}(A_{((\lambda+1),0^h)}, A_{((\lambda+1),0^h)})}{\mathrm{Den}(A_{(0^h)}, A_{(2^h)})/\mathrm{Den}(A_{(0^h)}, A_{(0^h)})} = q^{2h}.\end{aligned}$$

This implies that

$$\begin{aligned}\widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{H}}(x) &= q^{2h}q^{-h-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} \\ &= q^{h-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.\end{aligned}$$

Now, by Theorem 8.19, Theorem 8.21, we have that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) = \widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) - \widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{H}}(x) = \begin{cases} 0 & \text{if } \mathrm{val}((x,x)) \leq -2, \\ -\dfrac{1}{q^h} D_{n-1,h-1}(a,b,c) & \text{if } \mathrm{val}((x,x)) = -1. \end{cases}$$

This proves (1) and (2) when $(a,b,c) = (1,h,n-h-2)$.

Assume that $(a,b,c) = (1,h-2,n-h)$ and let us write $L'^{\flat} = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are π^λ $(\lambda \geq 2)$, πI_{h-2}, I_{n-h}, respectively. Then, by Proposition 2.6, $\mathcal{Z}(L'^{\flat})^\circ_{\mathscr{H}}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ in $\mathcal{N}_h^{[h]}$. Also, in $\mathcal{N}_h^{[h]}$, we have that $\mathcal{Z}(w) = \mathcal{Y}(\pi^{-1}w)$ for any $w \in \mathbb{V}$, and hence $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ can be reduced to $\mathcal{Z}(L_2)^\circ_{\mathscr{H}} = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[2]}$. Now, note that

by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[2]}$, and hence $\mathrm{Int}_{L_2^\circ,\mathscr{H}}(x) = \mathrm{Int}_{L_2^\circ}(x) = \partial\mathrm{Den}_{L_2^\circ}^{2,2}(x)$.

By Theorem 8.19 and Theorem 8.21, we have that

$$\widehat{\mathrm{Int}}_{L_2^\circ}(x) = \widehat{\partial\mathrm{Den}}_{L_2^\circ}^{2,2}(x) = \begin{cases} q^{-2}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } \mathrm{val}((x,x)) \le -2, \\ q^{-3}(q+1)(1-q^{-2})^{-1} - q^{-2} & \text{if } \mathrm{val}((x,x)) = -1. \end{cases}$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[2]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(\pi^{-1}L_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^{h-2}}\widehat{\mathrm{Int}}_{L_2^\circ}(x) = \begin{cases} q^{-h}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} & \text{if } \mathrm{val}((x,x)) \le -2, \\ q^{-h-1}(q+1)(1-q^{-2})^{-1} - q^{-h} & \text{if } \mathrm{val}((x,x)) = -1. \end{cases}$$

Now, by Theorem 8.19, Theorem 8.21, we have that

$$\widehat{\partial\mathrm{Den}}_{L'^\flat\circ,\mathscr{V}}^{n,h}(x) = \widehat{\partial\mathrm{Den}}_{L'^\flat\circ}^{n,h}(x) - \widehat{\mathrm{Int}}_{L'^\flat\circ,\mathscr{H}}(x) = \begin{cases} 0 & \text{if } \mathrm{val}((x,x)) \le -2, \\ -\dfrac{1}{q^h}D_{n-1,h-1}(a,b,c) + \dfrac{1}{q^h} & \text{if } \mathrm{val}((x,x)) = -1. \end{cases}$$

This proves (1) and (2) when $(a,b,c) = (1, h-2, n-h)$.

Assume that $(a,b,c) = (0, h-1, n-h)$ and let us write $L'^\flat = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are πI_1, πI_{h-2}, I_{n-h}, respectively. Note that in this case, $\mathrm{val}((x,x)) \neq -1$ since $\mathrm{val}(L'^\flat \oplus \langle x\rangle) \equiv h+1 \pmod 2$. By Proposition 2.6, $\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ in $\mathcal{N}_h^{[h]}$. Also, in $\mathcal{N}_h^{[h]}$, we have that $\mathcal{Z}(w) = \mathcal{Y}(\pi^{-1}w)$ for any $w \in \mathbb{V}$, and hence $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ can be reduced to $\mathcal{Z}(L_2)^\circ_{\mathscr{H}} = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[2]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[2]}$, and hence

$$\mathrm{Int}_{L_2^\circ,\mathscr{H}}(x) = \mathrm{Int}_{L_2^\circ}(x) = \partial\mathrm{Den}_{L_2^\circ}^{2,2}(x).$$

By Theorem 8.19, we have that

$$\widehat{\mathrm{Int}}_{L_2^\circ}(x) = \widehat{\partial\mathrm{Den}}_{L_2^\circ}^{2,2}(x) = q^{-1}\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[2]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(\pi^{-1}L_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$ is

$$\frac{1}{q^{h-2}}\widehat{\mathrm{Int}}_{L_2^\circ}(x) = q^{-h+1}\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

By Theorem 8.19 we have that $\widehat{\partial\mathrm{Den}}_{L'^\flat\circ,\mathscr{V}}^{n,h}(x) = \widehat{\partial\mathrm{Den}}_{L'^\flat\circ}^{n,h}(x) - \widehat{\mathrm{Int}}_{L'^\flat\circ,\mathscr{H}}(x) = 0$. This proves (1) when $(a,b,c) = (0, h-1, n-h)$.

Assume that $(a,b,c) = (0, h, n-h-1)$ and let us write $L'^\flat = L_1 \oplus L_0$, where the hermitian matrix of L_1, L_0 are πI_h, I_{n-h-1}, respectively. By Proposition 2.6, $\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_1)^\circ_{\mathscr{H}}$ in $\mathcal{N}_{h+1}^{[h]}$. Therefore, by Theorem 3.3, we have that $\mathcal{Z}(L_1)^\circ_{\mathscr{H}} = \mathcal{Y}(\pi^{-1}L_1)$ and hence $\langle \mathcal{Y}(\pi^{-1}L_1), \mathcal{Z}(x)\rangle$ is $\langle \mathcal{Z}(x)\rangle$ in $\mathcal{N}_1^{[0]}$. Now, By [KR11, Theorem 1.1] and Proposition 2.6, we know that $\langle \mathcal{Z}(x)\rangle = \partial\mathrm{Den}_{\emptyset}^{1,0}(x)$. By Theorem 8.19 and Theorem 8.21, we have that

$$\widehat{\partial\mathrm{Den}}_{\emptyset}^{1,0}(x) = \mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_1^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Y}(\pi^{-1}L_1)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$is

$$\frac{1}{q^h}\widehat{\partial\mathrm{Den}}^{1,0}_{\emptyset}(x) = q^{-h}\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

This implies that

$$\widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{H}}(x) = q^{-h}\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}$$

Now, by Theorem 8.19, Theorem 8.21, we have that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) = \widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) - \widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{H}}(x) = \begin{cases} 0 & \text{if } \mathrm{val}((x,x)) \le -2, \\ -\dfrac{1}{q^h} D_{n-1,h-1}(a,b,c) & \text{if } \mathrm{val}((x,x)) = -1. \end{cases}$$

This proves (1) and (2) when $(a,b,c) = (0,h,n-h-1)$.

When $(a,b,c) = (0,h+1,n-h-2)$, let us write $L'^\flat = L_2 \oplus L_1 \oplus L_0$, where the hermitian matrix of L_2, L_1, L_0 are $\pi^1 I_1$, πI_h, I_{n-h-2}, respectively. Note that in this case, $\mathrm{val}((x,x)) \neq -1$ since $\mathrm{val}(L'^\flat \oplus \langle x\rangle) \equiv h+1 \pmod 2$. By Proposition 2.6, $\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}}$ in $\mathcal{N}_n^{[h]}$ can be reduced to $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ in $N_{h+2}^{[h]}$. Therefore, by Theorem 3.3, we have

$$\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}} = \sum_{\substack{L_2 \oplus L_1 \subset L_2 \oplus N \subset \pi^{-1}(L_2 \oplus L_1) \\ N \simeq (\pi^{-1})^h}} \mathcal{Z}(L_2)^\circ \cdot \mathcal{Y}(N)^\circ. \tag{9.5}$$

By Proposition 2.6 again, $\mathcal{Z}(L_2 \oplus L_1)^\circ_{\mathscr{H}}$ can be reduced to $\mathcal{Z}(L_2)^\circ_{\mathscr{H}} = \mathcal{Z}(L_2)^\circ$ in $\mathcal{N}_2^{[0]}$. Now, note that by [KR11, Theorem 1.1], we know that the Kudla-Rapoport conjecture holds in the case of $\mathcal{N}_2^{[0]}$, and hence $\mathrm{Int}_{L_2^\circ,\mathscr{H}}(x) = \mathrm{Int}_{L_2^\circ}(x) = \partial\mathrm{Den}^{2,0}_{L_2^\circ}(x)$.

By Theorem 8.19 and Theorem 8.21, we have that

$$\widehat{\mathrm{Int}}_{L_2^\circ}(x) = \widehat{\partial\mathrm{Den}}^{2,0}_{L_2^\circ}(x) = q^{-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by [Zha22, Proposition 8.2], we know that via the embedding $\mathcal{N}_2^{[0]} \hookrightarrow \mathcal{N}_n^{[h]}$, the Fourier transform of $\chi(\mathcal{N}_n^{[h]}, O_{\mathcal{Z}(L_2)^\circ} \otimes^{\mathbb{L}} O_{\mathcal{Y}(N_2)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(L_0)} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})$is

$$\frac{1}{q^h}\widehat{\mathrm{Int}}_{L_2^\circ}(x) = q^{-h-1}(q+1)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by (1.6) and Lemma 8.22, the number of lattices $L_2 \oplus N$ in the sum in (9.5) is

$$\begin{aligned} &\frac{\mathrm{Den}(A_{(1,(-1)^h)}, A_{(1,1^h)})/\mathrm{Den}(A_{(1,(-1)^h)}, A_{(1,(-1)^h)})}{\mathrm{Den}(A_{((-1)^h)}, A_{(1^h)})/\mathrm{Den}(A_{((-1)^h)}, A_{((-1)^h)})} \\ &= \frac{\mathrm{Den}(A_{(2,0^h)}, A_{(2,2^h)})/\mathrm{Den}(A_{(2,0^h)}, A_{(2,0^h)})}{\mathrm{Den}(A_{(0^h)}, A_{(2^h)})/\mathrm{Den}(A_{(0^h)}, A_{(0^h)})} = q^{2h}\frac{1-(-q)^{-h-1}}{1-(-q)^{-1}}. \end{aligned}$$

This implies that

$$\widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{H}}(x) = q^h(1-(-q)^{-h-1})\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1}.$$

Now, by Theorem 8.19, Theorem 8.21, we have that

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x) = \widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x) - \widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{H}}(x) = 0.$$

This proves (1) when $(a,b,c) = (0,h+1,n-h-2)$. This finishes the proof of the theorem. □

Proposition 9.3. *(cf.* [LZ22a, Proposition 7.3.4], [LL22, Proposition 2.22]*) Assume that $L^\flat$ be an O_F-lattice of rank $n-1$ in $\mathbb{V}$. Then, $\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(x)$ extends uniquely to a compactly supported locally constant function on $\mathbb{V}$ (we still denote it by $\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(x)$).*

Proof. Even though our situation is more similar to [LZ22a, Proposition 7.3.4], we do not have a functional equation like [LZ22a, (3.2.0.2)] yet. Therefore, let us follow the proof of [LL22, Proposition 2.22]. Basically, the proof is very similar to the proof of Theorem 9.2 (1).

Note that if $L^\flat$ is not integral, then $\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(x)=0$. Therefore, we can assume that $L^\flat$ is integral. Now, it suffices to show that for every $y\in L_F^\flat/L^\flat$, there exists an integer $\delta(y)>0$ such that $\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(y+x)$ is constant for $x\in\pi^{\delta(y)}((L_F^\flat)^\perp)^{\geq 0}\backslash\{0\}$, where $\mathbb{V}=L_F^\flat\oplus(L_F^\flat)^\perp$ and $((L_F^\flat)^\perp)^{\geq 0}=\{(L_F^\flat)^\perp\mid (x,x)\in O_F\}$. If $L^\flat+\langle y\rangle$ is not integral, then there exists $\delta(y)$ such that $L^\flat+\langle y+x\rangle$ is not integral for $x\in\pi^{\delta(y)}((L_F^\flat)^\perp)^{\geq 0}\backslash\{0\}$, and hence $\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(y+x)=0$. Therefore, we can assume that $L^\flat+\langle y\rangle$ is integral. Let $(a_1,a_2,\dots,a_{n-1})$ $(a_1\leq a_2\cdots\leq a_{n-1})$ be the fundamental invariants of $L^\flat$ and let $\delta(y)=a_{n-1}+2$. Then, it suffices to show that for a fixed pair (f_1,f_2) of generators of $((L_F^\flat)^\perp)^{\geq 0}$, we have

$$\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(y+\pi^\delta f_1)-\partial\mathrm{Den}^{n,h}_{L'^\flat,\mathscr{V}}(y+\pi^{\delta-1}f_2)=0,$$

for $\delta>\delta(y)=a_{n-1}+2$.

Now, let us introduce some notations from [LL22, Lemma 2.24]. Let $\mathfrak{L}$ be the set of O_F-lattices in $\mathbb{V}$ containing $L^\flat$, and let $\mathfrak{C}$ be the set of triples $(L'^\flat,\delta,\epsilon)$ such that $L'^\flat$ is an O_F-lattice of $L_F^\flat$ containing $L^\flat$, $\delta\in\mathbb{Z}$, and $\epsilon:\pi^\delta((L_F^\flat)^\perp)^{\geq 0}\to L'^\flat\otimes_{O_F}F/O_F$ is an O_F-linear map. Then, consider the map $\theta:\mathfrak{L}\to\mathfrak{C}$ sending L to $(L\cap L_F^\flat,\delta_L,\epsilon_L)$ where δ_L is the maximal integer such that the image of L under the projection $\mathrm{Pr}_\perp:\mathbb{V}\to(L_F^\flat)^\perp$ is contained in $\pi^{\delta_L}((L_F^\flat)^\perp)^{\geq 0}$, and ϵ_L is the extension map $\pi^{\delta_L}((L_F^\flat)^\perp)^{\geq 0}\to L'^\flat\otimes_{O_F}F/O_F$ induced by the short exact sequence

$$0\to L\cap L_F^\flat\to L\to\pi^{\delta_L}((L_F^\flat)^\perp)^{\geq 0}\to 0.$$

Then, as in the proof of [LL22, Lemma 2.24 (1)], θ is a bijection and its inverse is given by sending $(L'^\flat,\delta,\epsilon)$ to the O_F-lattice L generated by $L'^\flat$ and $\epsilon(x)+x$ for every $x\in\pi^\delta((L_F^\flat)^\perp)^{\geq 0}$.

Now, for every $\delta'\in\mathbb{Z}$, we define the following sets

$$\begin{aligned}\mathfrak{L}_1^{\delta'}&:=\{L\in\mathfrak{L}\mid L\subset L^\vee,\delta_L=\delta',y+\pi^\delta f_1\in L\},\\ \mathfrak{L}_2^{\delta'}&:=\{L\in\mathfrak{L}\mid L\subset L^\vee,\delta_L=\delta',y+\pi^{\delta-1}f_2\in L\},\end{aligned}$$

and for $L^\flat\subset L'^\flat$, we define

$$\begin{aligned}\mathfrak{L}_{1,L'^\flat}^{\delta'}&:=\{L\in\mathfrak{L}\mid L\subset L^\vee,\delta_L=\delta',y+\pi^\delta f_1\in L,L\cap L_F^\flat=L'^\flat\},\\ \mathfrak{L}_{2,L'^\flat}^{\delta'}&:=\{L\in\mathfrak{L}\mid L\subset L^\vee,\delta_L=\delta',y+\pi^{\delta-1}f_2\in L,L\cap L_F^\flat=L'^\flat\}.\end{aligned}$$

Since $\partial\mathrm{Den}^{n,h}_{L^\flat,\mathscr{V}}(x)$ is a certain sum of $\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x)$, it suffices to show that

$$\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(y+\pi^\delta f_1)-\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(y+\pi^{\delta-1}f_2)=0,$$

for $\delta>\delta(y)=a_{n-1}+2$ and $L^\flat\subset L'^\flat$.

By definition, we have that

$$\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(y+\pi^{\delta}f_1)=\sum_{\delta'\leq\delta}\sum_{L\in\mathfrak{L}^{\delta'}_{1,L'^{\flat}}}D_{n,h}(L),$$
$$\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(y+\pi^{\delta-1}f_2)=\sum_{\delta'\leq\delta-1}\sum_{L\in\mathfrak{L}^{\delta'}_{2,L'^{\flat}}}D_{n,h}(L).$$

Therefore, it suffices to show that

$$\sum_{\delta'\leq\delta}\sum_{L\in\mathfrak{L}^{\delta'}_{1,L'^{\flat}}}D_{n,h}(L)-\sum_{\delta'\leq\delta-1}\sum_{L\in\mathfrak{L}^{\delta'}_{2,L'^{\flat}}}D_{n,h}(L)=0, \tag{9.6}$$

for all $\delta>\delta(y)=a_{n-1}+2$ and $L^{\flat}\subset L'^{\flat}$.

Since $\delta>a_{n-1}+2$, we have that for $\delta'\leq 2$, we have

$$\mathfrak{L}^{\delta'}_{1,L'^{\flat}}=\mathfrak{L}^{\delta'}_{2,L'^{\flat}}=\{L\in\mathfrak{L}\mid L\subset L^{\vee},\delta_L=\delta',y\in L,L\cap L_F^{\flat}=L'^{\flat}\}.$$

Therefore, (9.6) equals to $\sum_{\delta'=2}^{\delta}\sum_{L\in\mathfrak{L}^{\delta'}_{1,L'^{\flat}}}D_{n,h}(L)-\sum_{\delta'=2}^{\delta-1}\sum_{L\in\mathfrak{L}^{\delta'}_{2,L'^{\flat}}}D_{n,h}(L)=0.$

Also, the automorphism of $\mathfrak{C}$ sending $(L'^{\flat},\delta',\epsilon)$ to $(L'^{\flat},\delta'-1,\epsilon(\pi\alpha\cdot))$, where $\alpha\in O_F^{\times}$, $f_1=\alpha f_2$, induces a bijection from $\mathfrak{L}^{\delta'}_{1,L'^{\flat}}$ to $\mathfrak{L}^{\delta'-1}_{2,L'^{\flat}}$. Therefore, it suffices to show that

$$\sum_{L\in\mathfrak{L}^{2}_{1,L'^{\flat}}}D_{n,h}(L)=1_{L'^{\flat}}(y)\sum_{L\subset L^{\vee},L\cap L_F^{\flat}=L'^{\flat},\delta_L=2}D_{n,h}(L)=0. \tag{9.7}$$

Note that by Theorem 9.2 (1), we have that for $\mathrm{val}((x,x))\leq -2$,

$$\widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat\circ}}(x)=\mathrm{vol}(L'^{\flat})\sum_{L'^{\flat}\subset L\subset L^{\vee},L\cap L_F^{\flat}=L'^{\flat},x\in L^{\vee}}D_{n,h}(L)\,\mathrm{vol}(\mathrm{Pr}_{\perp}(L))=0.$$

Now, choose x to be generators of $\pi^{-2}((L_F^{\flat})^{\perp})^{\geq 0}$ and $\pi^{-3}((L_F^{\flat})^{\perp})^{\geq 0}$ and then take the difference. Then, we have that

$$\sum_{L\subset L^{\vee},L\cap L_F^{\flat}=L'^{\flat},\delta_L=2}D_{n,h}(L)=0.$$

This shows that (9.7) holds which finishes the proof of the proposition. □

Theorem 9.4. *Assume that Conjecture 4.3 above holds for $\mathcal{N}_n^{[h]}$ and Conjecture 5.6 holds for $\mathcal{Z}$-cycles in $\mathcal{N}_{n-1}^{[h-1]}$. Then, Conjecture 5.6 holds for $\mathcal{Z}$-cycles in $\mathcal{N}_n^{[h]}$.*

Proof. As in [LZ22a, section 8.2], we will prove this inductively. Let $L^{\flat}\subset\mathbb{V}$ be a rank $n-1$ lattice such that $L_F^{\flat}$ is non-degenerate, and let $x\in\mathbb{V}\backslash L_F^{\flat}$. By definition of $\partial\mathrm{Den}^{n,h}_{L^{\flat},\mathscr{V}}(x)$ and $\partial\mathrm{Den}^{n,h}_{L^{\flat\circ},\mathscr{V}}(x)$, it suffices to show that

$$\partial\mathrm{Den}^{n,h}_{L^{\flat},\mathscr{V}}(x)=\mathrm{Int}_{L^{\flat},\mathscr{V}}(x), \tag{9.8}$$

or equivalently

$$\partial\mathrm{Den}^{n,h}_{L'^{\flat\circ},\mathscr{V}}(x)=\mathrm{Int}_{L'^{\flat\circ},\mathscr{V}}(x), \tag{9.9}$$

for all $L^{\flat}\subset L'^{\flat}\subset L'^{\flat\vee}$.

Now, assume that (9.8) holds for $L'^\flat$ such that $\mathrm{val}(L'^\flat) < \mathrm{val}(L^\flat)$.

Let $(a_1, a_2, \dots, a_{n-1})$ $(0 \le a_1 \le \cdots \le a_{n-1})$ be the fundamental invariants of $L^\flat$. Let $M = M(L^\flat) = L^\flat \ominus \langle u \rangle$ for some $u \in \mathbb{V}$ such that $\mathrm{val}((u,u)) = a_n \coloneqq a_{n-1}$ or $a_{n-1}+1$, so that

$$a_1 + a_2 + \cdots + a_n \equiv h + 1 \pmod 2.$$

Now, assume that $(a_1', a_2', \dots, a_n')$ be the fundamental invariants of the lattice $L^\flat + \langle x \rangle$ with a basis $(e_1', \dots, e_n')$ such that $\mathrm{val}(e_i', e_i') = a_i'$. Let $L'^\flat = \langle e_1', \dots, e_{n-1}' \rangle$ ane let $x' = e_n'$. Then, we have that

$$\mathrm{Int}_{L^\flat}(x) = \mathrm{Int}_{L'^\flat}(x), \quad \partial\mathrm{Den}^{n,h}_{L^\flat}(x) = \partial\mathrm{Den}^{n,h}_{L'^\flat}(x).$$

By [LZ22a, Lemma 8.2.2], if $x \notin M$, then $\mathrm{val}(L'^\flat) < \mathrm{val}(L^\flat)$, and hence by the inductive hypothesis, we have that

$$\mathrm{Int}_{L'^\flat}(x) = \partial\mathrm{Den}^{n,h}_{L'^\flat}(x).$$

This implies that the support of

$$\phi = \mathrm{Int}_{L^\flat,\mathscr{V}}(x) - \partial\mathrm{Den}^{n,h}_{L^\flat,\mathscr{V}}(x) \in C_c^\infty(\mathbb{V})$$

is contained in the lattice M. Here, $\mathrm{Int}_{L^\flat,\mathscr{V}}(x) - \partial\mathrm{Den}^{n,h}_{L^\flat,\mathscr{V}}(x)$ is in $C_c^\infty(\mathbb{V})$ by Proposition 9.3, Theorem 4.5, [LZ22a, Lemma 6.2.1], and [San17, Lemma 2.11].

Now, let us consider x such that $\mathrm{val}((x,x)) < 0$ and $x \perp L^\flat$. Since we have assumed that Conjecture 4.3 below holds, by [LZ22a, Lemma 6.3.1] and [Zha22, Theorem 8.1], we have that

$$\widehat{\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Z}(x)})} = \widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}}(x) = -\frac{1}{q^h}\mathrm{Int}_{L^\flat,\mathscr{V}}(\mathcal{Y}(x)) \coloneqq -\frac{1}{q^h}\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}).$$

From now on, we use $\mathrm{Int}_{L^\flat,\mathscr{V}}(\mathcal{Y}(x))$ (resp. $\mathrm{Int}_{L^{\flat\circ},\mathscr{V}}(\mathcal{Y}(x))$) to denote $\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)})$ (resp. $\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)})$).

Furthermore, note that we can decompose this into primitive parts

$$-\frac{1}{q^h}\mathrm{Int}_{L^\flat,\mathscr{V}}(\mathcal{Y}(x))) = \sum_{L^\flat \subset L'^\flat \subset L'^{\flat\vee}} -\frac{1}{q^h}\mathrm{Int}_{L'^{\flat\circ},\mathscr{V}}(\mathcal{Y}(x)) = \sum_{L^\flat \subset L'^\flat \subset L'^{\flat\vee}} \widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{V}}(x).$$

Now, let us compare this with the analytic side.

When $\mathrm{val}((x,x)) \le -2$, we have that $\mathcal{Y}(x)$ is empty. Therefore, combining this with Theorem 9.2, we have that

$$\widehat{\phi}(x) = \widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}}(x) - \widehat{\partial\mathrm{Den}}^{n,h}_{L^\flat,\mathscr{V}}(x) = 0 - 0 = 0.$$

When $\mathrm{val}((x,x)) = -1$, by Proposition 2.6, we have that

$$\begin{aligned}\widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{V}}(x) = -\frac{1}{q^h}\mathrm{Int}_{L^{\flat\circ},\mathscr{V}}(\mathcal{Y}(x)) &= -\frac{1}{q^h}\chi(\mathcal{N}_n^{[h]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{V}} \otimes^{\mathbb{L}} O_{\mathcal{Y}(x)}) \\ &= -\frac{1}{q^h}\chi(\mathcal{N}_{n-1}^{[h-1]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ - {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}}).\end{aligned}$$

Since we have assumed that Conjecture 5.6 holds for $\mathcal{N}_{n-1}^{[h-1]}$, we have that

$$-\frac{1}{q^h}\chi(\mathcal{N}_{n-1}^{[h-1]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ) = -\frac{1}{q^h}D_{n-1,h-1}(a,b,c),$$

where $(a,b,c) = (t_{\ge 2}(\lambda), t_1(\lambda), t_0(\lambda))$ and $\lambda \in \mathcal{R}_{n-1}^{0+}$ is the fundamental invariants of $L'^\flat$.

For $\chi(\mathcal{N}_{n-1}^{[h-1]}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ_{\mathscr{H}})$, we have the following two cases.

First, assume that $(a,b,c) \neq (1,h-2,n-h)$. Then, by Theorem 3.3 (or see the proof of Theorem 9.2), we have that $\mathcal{Z}(L'^{\flat})^{\circ}_{\mathscr{H}}$ is empty or the sum of $\mathcal{Z}(L)^{\circ} \cdot \mathcal{Y}(N)$ for some lattice $N \simeq (\pi^{-1})^h$. Since $\mathcal{Y}(N)$ is empty in $\mathcal{N}_{n-1}^{[h-1]}$, we have that $\chi(\mathcal{N}_{n-1}^{[h-1]}, {}^{\mathbb{L}}\mathcal{Z}(L'^{\flat})^{\circ}_{\mathscr{H}}) = 0$.

Now, assume that $(a,b,c) = (1,h-2,n-h)$. Then by Theorem 3.3 (or see the proof of Theorem 9.2), we have that

$$\mathcal{Z}(L'^{\flat})^{\circ}_{\mathscr{H}} = \mathcal{Z}(L_2)^{\circ} \cdot \mathcal{Y}(\pi^{-1}L_1) \cdot \mathcal{Z}(L_0),$$

where $L_2 \simeq \pi^{\lambda}$ $(\lambda \geq 2)$, $\pi^{-1}L_1 \simeq \pi^{-1}I_{h-2}$, and $L_0 \simeq I_{n-h}$. Therefore,

$$\chi(\mathcal{N}_{n-1}^{[h-1]}, {}^{\mathbb{L}}\mathcal{Z}(L'^{\flat})^{\circ}_{\mathscr{H}}) = \chi(\mathcal{N}_{1}^{[1]}, {}^{\mathbb{L}}\mathcal{Z}(L_2)^{\circ}) = D_{1,1}(1,0,0) = 1.$$

Combining these, we have that

$$\widehat{\mathrm{Int}}_{L'^{\flat\circ},\mathscr{V}}(x) = \begin{cases} -\dfrac{1}{q^h} D_{n-1,h-1}(a,b,c) & \text{if } (a,b,c) \neq (1,h-2,n-h), \\ -\dfrac{1}{q^h} D_{n-1,h-1}(a,b,c) + \dfrac{1}{q^h} & \text{if } (a,b,c) = (1,h-2,n-h). \end{cases}$$

Therefore, by Theorem 9.2, we have that

$$\widehat{\mathrm{Int}}_{L'^{\flat},\mathscr{V}}(x) - \widehat{\partial\mathrm{Den}}^{n,h}_{L'^{\flat},\mathscr{V}}(x) = 0 \text{ for } \mathrm{val}((x,x)) < 0.$$

This implies that $\widehat{\phi} = 0$ for $\mathrm{val}((x,x)) < 0$.

Now, we only need to follow the proof of [LZ22a, Theorem 8.2.1].

Since ϕ is invariant under $L^{\flat}$ and $\mathrm{Supp}(\phi) \subset M$, we have that

$$\phi = 1_{L^{\flat}} \otimes \phi_{\perp},$$

where $\phi_{\perp} \in C_c^{\infty}((L_F^{\flat})^{\perp})$ and $\phi_{\perp}$ is supported on $M_{\perp} = \langle u \rangle$. Then, we have

$$\widehat{\phi} = \mathrm{vol}(L^{\flat}) 1_{L^{\flat\vee}} \otimes \widehat{\phi_{\perp}},$$

and $\widehat{\phi_{\perp}}$ is invariant under the translation by $\langle u^{\vee} \rangle = \langle \pi^{-a_n} u \rangle$. Since $\mathrm{val}((u^{\vee}, u^{\vee})) = -a_n < 0$ and $\widehat{\phi_{\perp}} = 0$ for every x such that $x \perp L^{\flat}$, $\mathrm{val}((x,x)) < 0$, we have that $\widehat{\phi_{\perp}}$ vanishes identically and hence $\phi = 0$. This finishes the proof of the Theorem. □

Theorem 9.5. *(cf.* [LZ22a, Theorem 8.2.1, Theorem 10.5.1]*) Conjecture 5.6 holds for $\mathcal{Z}$-cycles in $\mathcal{N}_n^{[0]}, \mathcal{N}_n^{[1]}, \mathcal{N}_n^{[n-1]}, \mathcal{N}_n^{[n]}$, and $\mathcal{N}_4^{[2]}$.*

Proof. This follows from Theorem 9.4 and Theorem 4.5. □

Remark 9.6. We remark that the Kudla-Rapoport conjecture for $\mathcal{N}_n^{[0]}$, $\mathcal{N}_n^{[n]}$ (good reductions case), $\mathcal{N}_n^{[1]}$ (almost self-dual case) is already proved in [LZ22a]. Therefore, our new cases are $\mathcal{N}_n^{[n-1]}$ and $\mathcal{N}_4^{[2]}$. We also remark that our work gives a different proof of the conjecture for $\mathcal{N}_n^{[1]}$.

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